

1. Sequences and Recursion

2. You should be familiar with functions and function notation.

In this lesson, we will define sequences by using explicit formulas and using recursive formulas

3. A sequence is an ordered set of numbers, there is a first element of the sequence, followed by a second, third and so on. The  $n$ th term of the sequence is denoted  $a_n$ .

4. Typically, a sequence is an infinite list.

5. For example, the positive even numbers form an increasing sequence, 2, 4, 6, 8, 10 and so on.

6. (a) We often denote the entire sequence with a symbol. We can let  $E$  be the sequence of positive even integers.

(b) We usually denote the members of the sequence with lower case letters corresponding to the name of the sequence. In this case  $e_1$  is the first term in the sequence, which is 2.

(c) Sometimes it is easy to tell the pattern and be able to skip ahead several terms. Suppose we wished to find the 24th term. Each term is two larger than the previous term, so the 24th term is 24 times 2, or 48.

(d) In this case, we can write an explicit formula. We get the  $n$ th term by multiplying  $n$  by 2, so  $e_n = 2n$ .

(e) We can also denote the entire sequence by using the lower case letter and subscript notation.

7. Notice the similarities between sequences defined explicitly and function notation. For sequences, we usually use the subscript  $n$  as the input, whereas functions typically use  $x$  as the input variable. Sequence names commonly use many different letters, just as many different letters are used to represent functions. Trigonometry uses three letter abbreviations for functions like sine and cosine. Note that in this case, the explicit formula for the sequence is similar to the equation of a line. The difference between sequences and functions, is that a function commonly allows all real numbers as inputs, whereas sequences are ordered using the inputs 1, 2, 3, etc.

8. (a) Here is another example using an explicit definition. To find the first term,  $m_1$ , we plug 1 into the formula  $2^1 - 1 = 1$ .

(b) Plugging in 2, we get  $2^2 - 1 = 3$ .

(c) Plugging in 3, we get  $2^3 - 1 = 7$ .

(d) Continuing on, we get 15 as the fourth term.

(e) and 31 as the fifth.

(f) .

9.
  - (a) We can also define a sequence recursively. To do so, we must define the first element, and then give a recursive formula to find future elements based on the previous elements.
  - (b) In the case of the positive even numbers, the first term is 2. We get the next even number by adding two to the previous number.
  - (c) We can write the recursive part of the formula using subscripts. To get term  $n + 1$ , we take term  $n$  and add 2.
10.
  - (a) A commonly used recursive formula is that of compounding interest. We have an initial deposit, and then each year, we accrue interest.
  - (b) To get the amount of money after 1 year, we multiply our initial invest of \$100 by 6% earning \$6 of interest, which is added to the \$100 to give us \$106. We then add 6% of \$106 to get our account balance at the end of year two, etc.
  - (c) We can get the same result by multiplying by 1.06. The 1 represents the fact that we get to keep the money we already have in the bank, the .06 represents the interest.
  - (d) Sometimes, it makes more sense to define the sequence starting with the subscript 0, rather than 1.
  - (e) To write an explicit formula, we multiply our initial deposit of \$100 by 1.06 for each of  $n$  years.
  - (f) It is often easier to use the explicit formula to calculate terms much later in the sequence.
  - (g) In the recursive definition, we would need to repeatedly multiply by 1.06 a total of 50 times. With the explicit definition, we can get the answer with one calculation.
11.
  - (a) Here is another example of a recursively defined sequence. The first element is 1. To get the next element, we multiply the previous element by 3, then subtract 1. The second element is  $3(1) - 1 = 2$ .
  - (b) The third element is  $3(2) - 1 = 5$ . The fourth element is  $3(5) - 1 = 14$ ,  $3(14) - 1 = 41$  and so on.
  - (c) It is not always easy to discover an explicit formula.
12. Sometimes, a recursive formula can depend on two or more previous terms in the sequence. A famous sequence is the Fibonacci sequence. To get the third term, we add the first two terms. To get the 4th term, we add to two previous terms.
13. To get terms to alternate in an explicit definition, we can use  $(-1)^n$ . When  $n$  is even, we will get positive 1, and when  $n$  is odd, we will get -1. This will cause the terms to alternate in sign.
14. To recap: An explicit definition gives a formula to calculate each term independently, much like a function. In a recursive definition, we need to specify the beginning of the sequence, and a method to determine later terms of the sequence, based on earlier terms of the sequence.