1. Arithmetic Sequences and Series

2. You should be familiar with sequences of numbers, defined explicitly and recursively, and should be familiar with Sigma Notation. It will also be helpful to know the slope-intercept form of a line.

   In this lesson, we define a special kind of sequence, called an arithmetic sequence, and learn a technique to quickly find the sum of an arithmetic series.

3. An **arithmetic** sequence is a sequence of numbers in which the recursion is to add a constant, called the **common difference** This pattern is also called linear growth, the numbers grow at a constant amount each step. Much of the discussion of this topic is similar to equations of lines.

4. (a) For example, the sequence 5, 8, 11, 14 and so on is an arithmetic sequence. The first term is 5, and each step goes up by 3. To get the next term, take the previous term, and add 3.

   (b) We can find an explicit formula for an arithmetic sequence similar to the slope-intercept form of a line. Instead of having x as the input and y as the output in the equation $y = mx + b$, we have the subscript, n as the input, and the element of the sequence, a_n, as the output. We know the values when n is 1,2,3 and so on. The given information is equivalent to knowing that a line passes through the points $(1, 5)$, $(2, 8)$, $(3, 11)$ etc. The slope-intercept form of a line requires us to know the value when the input is zero.

   (c) In this case, we can go back 3 to get a starting value of 2. The 2 is not a member of the sequence. Think of 2 as the starting point, just as you think of the y-intercept as the starting point when graphing a line. The members of the sequence happen as we step forward.

   (d) The sequence is arithmetic, because as we step forward to the next term, the output changes by 3. The concept is the same as the slope of a line.

   (e) The sequence is equivalent to a line that has a slope of 3 and an intercept of 2. The formula for the output a_n is now the slope-intercept form of a line. $a_n = 3n + 2$, which is similar to $y = 3x + 2$

5. (a) Here is another example. The common difference is 4. Think of this sequence as a line whose slope is four. To get back to the starting point, subtract 4 from -3.

   (b) The starting point is -7, and then to get the sequence, we start taking steps of size 4 from -7.

   (c) The slope or common difference is 4, and the starting point is -7,

   (d) so the formula for $a_n = 4n - 7$. To get the 23rd term, simply plug 23 into the formula.

   (e) The 23rd term is 85.

6. (a) If we are given two terms, we can find the others by using the concept of slope. The 7th term is 22 and the 10th term is 31. To get from the 7th term to the 10th term, we take 3 steps and go up in value from 22 to 31, which is a gain of 9.
   (b) The common difference is the gain of 9 divided by the three steps, which is 3. This is the same formula as the slope formula for lines.
   (c) To get the starting point, we go back 7 steps of 3 from 22, so the starting point is 1.
   (d) The sequence goes up three each step, so the sequence begins 4,7,10,13. We can keep counting to verify that 22 is the 7th term and that 31 is the 10th term.
   (e) In general, the nth term is $3n + 1$
7. (a) Now let's try to find the sum of an arithmetic series. Suppose we wish to find the sum of the numbers from 1 to 100.
   (b) which could also be written in Sigma notation.
   (c) The trick is that instead of adding up the numbers in order from left to right, we want to pair them in a way that makes the addition easier.
   (d) By writing the series a second time in reverse order
   (e) we can then add the numbers vertically. $1 + 100 = 101$, $2 + 99 = 101$ etc. We now have 100 pairs of numbers that each add to 101.
   (f) So $2S = (101)(100)$
   (g) We can then divide by 2 to get the answer.
8. (a) Here is an example where we are adding the even numbers up to 100. We first need to know how many terms appear in the series.
   (b) There are 50 even numbers up to 100, so we are adding 50 terms, where the nth term equals $2n$.
   (c) We write the sum forward
   (d) and backward, and add vertically
   (e) We have 50 copies of 102, so $2S = (102)(50)$
   (f) Dividing by 2 gives us the answer.

9. (a) Here is one more example. One key step is to know how many terms we have. That is, what term in the sequence is the number 74?
   (b) We can see the common difference is 3.
   (c) The starting point is 2
   (d) So the formula for $a_n = 3n + 2$.
   (e) We now wish to find the value for n that gives the term 74.
   (f) Solving for n, we get $n = 24$, so we have 24 terms in the sequence.
   (g) We now wish to find the sum of 24 terms
   (h) We write the sum forwards and backwards and add vertically, getting 24 copies of 79.
   (i) Twice the sum is 79 times 24
   (j) So the sum is 948.

10. To recap: The formula for an arithmetic sequence is similar to the equation of a line. We need to find a starting point a_0 and a rate of change, called the common difference.

    To find the sum of an arithmetic series, write the sum twice, once forward, once backward, and sum vertically.