1. Solving Systems of Non-Linear Equations

2. You should be familiar with the graphs of conic sections like ellipses and parabolas and how to solve polynomial equations in one variable by factoring, completing the square and using the quadratic formula.

   In this lesson, we will find the points of intersection of polynomial equations.

3. Let's review the methods of solving polynomial equations. The first method is factoring. The key to this method is that factors multiply to zero only when one of the factors is zero. We need to have the right hand side of the equation equal to zero. We then factor the polynomial. There are certain patterns with which you may be familiar, but the basic method is simply to guess and check. On the second line, we need to begin with two terms that multiply to $4x^2$. It could be $4x$ and x, or $2x$ and $2x$. The last terms need to multiply to 14, again we have some choices, 1 and 14 or 2 and 7. and they can be mixed and matched in any order. We then try various combinations until we find the right one. Once we have the polynomial factored, set each factor equal to zero and solve.

4. Sometimes factoring may take more than one step. Here we first factor the fourth degree polynomial, then use the difference of squares to complete the factoring.

5. A second method makes use of completing the square. When one side is a perfect square, we can take the square root of both sides. Recall that we will get two answers, plus and minus.

6. If all else fails for quadratics, we can use the quadratic formula. Again, notice that we must have the right side of the equation equal to zero to begin.

7. One other fact before we start is the graph of $xy = 1$ which can be transformed into $y = 1/x$. This graph is a hyperbola which has been rotated $45°$ so the asymptotes are the x and y-axes.

8. Now let's solve equations. The methods are the same as we used for linear equations. The first method, substitution, relies on our ability to solve one equation for one variable, and substitute it into the other equation. Before we use algebraic techniques to solve, it is helpful to have a graph of the two equations to get a sense of how the two graphs might intersect. In this problem the first equation is a parabola and the second equation is a line, which may intersect in two points. Our solution should be two ordered pairs.

9. (a) To solve algebraically, we can substitute $2x^2$ for y in the second equation. We now have an equation in a single variable x, which we can solve.
   (b) First by getting one side of the equation equal to zero,
   (c) We can divide both sides by 2 to make the numbers smaller and easier to work with.
   (d) Then factor
   (e) and solve
   (f) Recall that our answers should be ordered pairs, so once we have found the x-values, we need to plug the x-values back into one of the original equations to find the y-values. Our points of intersection are $(2, 8)$ and $(-1, 2)$
   (g)

10. Here is a second example. It is a hyperbola of the second form with a circle. It is hard to tell how often these to graphs intersect.

11. (a) To solve algebraically, we can manipulate one equation to solve for one variable. In this case, dividing the first equation by x solves for y.
    (b) We can then plug this value for y into the second equation.
    (c) Simplify
    (d) Multiplying both sides by x^2 will eliminate fractions and make the equation easier to work with.
    (e) We get one side equal to zero.
    (f) Factor
    (g) Then find the values, first for x, then for y.
    (h) We have four points of intersection.

12. Here's an example of an ellipse and a hyperbola. A detailed graph will show that they won't intersect. Let's see what happens algebraically.

13. (a) We can use the elimination method. We can multiply the first equation by the constant 9, in order to cancel the y^2 terms. Combining the second and third lines
    (b) gives an equation in x.
    (c) Which we can solve by multiplying by 36 over 85, then reducing the fraction and taking a square root.
    (d) When we plug this answer back into the first equation, we find we can't solve for y.

14. (a) Sometimes we can substitute for something more complicated than a single variable. Here, we can solve for y^2 and substitute.
    (b) Plugging $x + 2$ into the first equation, gives an equation in x.
    (c) Get one side equal to zero.
    (d) Factor
    (e) Solve for x
    (f) Then find the possible y values

15. (a) Finally, we should verify that are answers really fit both equations. Occasionally in the solving process, we do something like squaring both sides that introduces incorrect solutions. This example is the top half of a parabola pointing to the right with a line. You may wish to draw a sketch to see that the graphs will intersect only once.
    (b) We can substitute.
    (c) Square both sides
    (d) Get one side equal to zero
    (e) Factor
    (f) and solve for x
    (g) Plugging these values into the second equation gives two answers. However, only one of them is correct. If we plug $x = 1$ into the first equation, we get $y = 1$, not -1. What happened is that on the third line, when $x = 1$, we have $1 = -1$, which is false, but when we squared both sides, we lose the negative sign. We need to discard the incorrect solution.

16. To recap: The techniques to solve equations are the same as with one variable, both the substitution and elimation methods may be useful to reduce the equation to one variable. We then solve the equation to find all the possibilities for one variable, substitute to find the values of the other variable to complete the ordered pairs. Knowledge of the graphs can help verify the answers.