1. Polynomial and Rational Inequalities

2. You should be familiar with graphing polynomial and rational functions. You should also be familiar with interval notation. In this lesson, we will solve inequalities involving polynomial and rational functions.

3. Recall the steps to analyze the graph of a rational function, in particular, we divided the number line at the x-intercepts, which occur when the numerator is zero, at the vertical asymptotes, which occur when the denominator is zero, then determined on which intervals the function was positive and negative. It was easiest to do this when the function was written in factored form.

4. Here is an example of a polynomial inequality.

   * First, we divide the number line where the expression on the left of the inequality equals 0. We can then examine on which intervals the function is positive and negative. To the right of 3, both factors are positive, so the function is positive.

   * Between -4 and 3, the factor $(x+4)$ stays positive, but the factor $(x-3)$ becomes negative. All together, the expression on the left is negative.

   * To the left of -4, both factors are negative and the the expression on the left is positive.

   * If we expand the expression on the left, it will be a polynomial that begins with x^2, this confirms our analysis, it is a parabola that opens upward.

   * The function is greater than or equal to zero when $x \leq -4$ and when $x \geq 3$, which we can write in interval notation.

5. Here is a similar example with exponents.

   * First, we divide the number line where $f(x) = 0$. To the right of 3, both factors are positive, so the function is positive.

   * Between -4 and 3, the factor $(x+4)$ stays positive. We cross past 3 for a factor with an even exponent, so we stay on the same side, the $(x-3)$ factor is a negative squared, and so remains positive.

   * To the left of -4, the cubed factor changes sign. Since it is an odd power, the graph will cross the axis to the negative side.

   * If we expand $f(x)$, it will be a polynomial that begins x^5, this confirms our analysis, an odd power of x begins on the negative side to the left, and ends on the positive side at the right.

   * The function is less than zero on the intervals when the function is negative, that is, when $x < -4$. We also need to include the points where the function equals zero, in other words, we include the points -4 and 3. The solution is the interval $x \leq -4$ plus the single point 3.

6. Here is an example where we need to simplify first. This technique only works when we are comparing a function to zero, so that we can analyze when the function is positive and

negative. Therefore, our first step is to make one side of the inequality zero, then write the function in factored form.

* We split the number line at the x-intercepts.

* And analyze the factors.

* The function is negative between -2 and 3.

* In this case, we are strictly less that 0, so we do not include the endpoints.

7. Here is the original example of a rational function. It is already in factored form. We place the x-intercepts and the vertical asymptotes. Note the the function equals zero at the x-intercepts so we will include these points, but the function is undefined at the vertical asymptotes, so we never include those points. We then analyze where the function is positive and negative. We find four intervals where the function is positive.

   * We also want to include points where the function is equal to zero. The function is undefined when $x = -1$ and when $x = 1$, so we cannot include those points. The function equals zero when $x = -3$ and 2, so we should include those points.

   * The rightmost two intervals can joined since they share an endpoint

8. To recap: To solve an inequality involving a rational function, first make sure that one side equals zero and that the rational expression is factored. Then analyze the number line by placing x-intercepts, vertical asymptotes and deciding where each factor is positive and negative. If the inequality is strictly $<$ or $>$, then the endpoints will not be included. If the inequality is $\geq$ or $\leq$, then the x-intercepts will be included in the intervals. The x-values of the vertical asymptotes are never included, since the function is undefined at these points. If two adjacent intervals share an endpoint, they can be combined into a single interval.