1. Graphing Polynomial Functions

2. You should be familiar with x- and y-intercepts, and factoring polynomials. In this lesson, we will sketch the graph of polynomial functions.

3. There are three key pieces of information that help us graph polynomial functions, the y-intercept, x-intercepts, and end behavior.

4. End behavior is the tendency of the graph to the far right and far left. In other words, to the right, x is very large and positive. The end behavior is a description of what is happening to the y-values when x goes far to the right, getting bigger and bigger. We can also look at what happens as x goes to the left, that is, when x is large and negative.

   * The concept and notation used in calculus is the idea of a limit, abbreviated 'l-i-m'. Below the limit is the tendency for the input variable. The arrow is read 'approaches', here we are interested in the limit as $x \to \infty$. That is, we are interested in what happens as x goes to the right. The quantity we are interested in comes to the right of the limit. Most often, we are interested in the function value $f(x)$. In this case, we see that the y-value is also growing very large, so this notation is stating that as x goes to the right, the function value, or y-value also gets large.

   * As x goes to the left, the y-value is going down fast.

5. It may be the case that a function approaches a specific value. Here is a graph of the exponential function. This is not a polynomial, but does illustrate the concept of end behavior. As the graph goes to the right, the values of the function e^x go up. As x goes to the left, the values of the function e^x get closer to zero.

6. Now let's graph a polynomial. Here is an example. We wish to graph $f(x)$. Some facts about the graph will be found easily from the expanded form.

   * Others can be found more easily from the factored form. One easy point to find is the y-intercept. Recall that the y-intercept occurs where $x = 0$, so to find the y-intercept, plug 0 in for x and calculate $f(x)$. In this case, use the expanded form to find that the y-intercept is -24.

   * The next step is to find the x-intercepts, which occur when the y-value or function value is zero. A polynomial will equal 0 only when one of the factors equals zero, so it is best to use the factored form. We get x-intercepts when $x = 4$, $x = -3$ and $x = -2$

   * Here the x-intercepts are placed. Finally, we look at the end behavior. The key is that when x is very large, only the leading term is significant. To illustrate the idea, suppose you were a billionaire and that $f(x)$ represented your wealth when $x = 1000$. If $x = 1000$, then the x^2 term is millions and the x^3 term is a billion. You start out with a billion dollars. Then you spend $24 for lunch. How much money do you have now? You're a billionaire, you probably don't waste much time worrying about how much lunch costs, you still have pretty close to a billion dollars. Then your Maserati needs engine work for $14,000. You probably

still don't worry too much about spending \$14,000, after all, your a billionaire. Ah, but there is good news, you just made one million dollars. Maybe you are mildly curious, but at the end of the day, when somebody asks you how much you are worth, you say 'I'm a billionaire'. It is only the leading term that has any noticeable effect when x is large, the rest of terms are insignificant. The importance to the graph is that when x is large, we can forget about everything else, and pretend the graph looks pretty much like $y = x^3$. Since it is an odd power, we know that it will go up on the right and down on the left.

* We can now put all the pieces together to sketch the graph. The graph goes down on the left, crosses through the intercepts, and finishes by going up on the right.

7. We can keep track of the end behavior and more in another way, by analyzing the factors. Each of the factors corresponds to an x-intercept. The x-intercepts are the points at which a factor changes from negative to positive. We can figure out whether $f(x)$ is positive or negative based on the factors. If we pick any number larger than 4, for example, 5, each of the factors is positive.

   * When we multiply three positive factors together, we get a positive answer.

   * Therefore, at the right of the graph, we will have positive y-values, and therefore the graph will be going up. Between -2 and 4, one of the factors changes sign from positive to negative. We can verify this by plugging in any number between -2 and 4, for example 0.

   * The first factor is negative, the others are positive, so the product will have one negative factor, and therefore is negative. The graph will be below the x-axis in this region.

   * Between -3 and -2, another factor changes sign, so we have two negative factors and one positive factor, making the product positive overall.

   * Finally, to the left, all the factors are negative, and the three negatives make the product negative. We now know that the graph will be below the x-axis starting at the left, above the x-axis between -3 and -2, below the x-axis between -2 and 4, and above the x-axis to the right of 4.

   * We can sketch the graph using this information, together with the x- and y-intercepts.

8. Here is another example. There are two places where the function equals zero. In other words, there are two x-intercepts. Notice that one of the factors is squared, and so in essence is two identical factors. By plugging zero in for x in the expanded form, we see that the y-intercept is -12.

   * If we choose a number larger than 2 for x, we see that $f(x)$ has its original negative sign, the $(x - 2)$ factor is positive, but recall that it really is two factors. The $(x + 3)$ term is also positive. All together, the product is negative.

   * If we choose a number between -3 and 2 for x, we see that $f(x)$ has its original negative sign, the two factors of $(x - 2)$ are now negative. The $(x + 3)$ term is still positive. All together, the product is negative.

* If we choose a value for x that is less than -3, we have the original negative sign, a double negative for the $(x-2)$ factors and a negative for the $(x+3)$ factor. All together, the product has four negative signs, and therefore is positive.

* We can now draw the graph, starting above the x-axis. Crossing below at $x=-3$, through the y-intercept of -12, then to the x-intercept at 2. This time, because of the double factor, we do not cross through. In essence, the double factor causes a double switch, from negative to positive then back again immediately to negative, so we finish going down.

9. Here is a third example. The x-intercepts are at -2, -1 and 1. To the right, all the factors are positive. As we move to the left of 1, the factor $(x-1)$ changes sign. Since it is raised to the fourth power, it adds four negatives. Any even amount of negative signs will cancel and remain positive, so when crossing an x-intercept associated with a factor raised to an even power, we will stay on the same side, in this case $f(x)$ remains positive and the graph stays above the x-axis. As we move to the left of -1, the factor $(x+1)$ changes sign. Since it is raised to the third power, it adds three negatives. Any odd amount of negative signs will leave a single negative, so when crossing an x-intercept associated with a factor raised to an odd power, we will cross to the other side, in this case $f(x)$ changes to negative and the graph crosses below the x-axis. As we move to the left of -2, the factor $(x+2)$ changes sign, and we cross back above the x-axis.

   * We sketch the graph, beginning above the x-axis on the left, crossing at -2 and again at -1, passing through the y-intercept of 2, returning to the x-intercept of 1, but staying positive to the right.

   Here is a screen shot of the graph from -2.2 to 1.7 using wolframalpha which shows the details of the intercepts. If we pan out to see the graph from -6 to 6, we see the end behavior that $f(x)$ looks like a graph of the leading term x^8, which is an even power of x.

10. To recap: To graph any polynomial, factor the polynomial to find the x-intercepts, plug in zero to find the y-intercept, then determine where the graph is above the x-axis and where it is below the x-axis in order to sketch the graph.