1. Graphing Rational Functions

2. You should be familiar with x- and y-intercepts, factoring polynomials and graphing polynomials. It may also help to know how to divide polynomials. In this lesson, we will sketch the graph of rational functions.

3. A rational function is the quotient of two polynomials. Here is an example.

4. There are four key piece that help us graph rational functions, the y-intercept, x-intercepts, vertical asymptotes and end behavior.

5. To illustrate the process, we begin with an example. We wish to graph $f(x)$. One easy point to find is the y-intercept.

   * Recall that the y-intercept occurs where $x = 0$, so to find the y-intercept, plug 0 in for x and calculate $f(x)$.

   * In this case, the y-intercept is at 108.

6. The next step is to find the x-intercepts, which occur when the y-value or function value is zero. A fraction will equal 0 only when the numerator equals zero, so we need to figure out when the numerator equals zero. This numerator is already factored, so all we need to do is set each factor equal to zero.

   * and solve.

   * The x-intercepts are at -3 and 2

7. We also wish to find where the function is undefined. This will happen when the denominator is zero. In this case, the denominator will equal zero when $x = -1$ and when $x = 1$

   * We can now proceed to analyze the intervals on which the function is positive and negative similar to the method for polynomials. We include the x-intercepts

   * the function is undefined when $x = 1$ or -1. When we are near either value, the numerator will be large and the denominator will be very close to zero. This ratio will be large, the graph will tend to infinity near these values, therefore, there will be an asymptote at $x = 1$ and at $x = -1$.

   * Now that we have divided the real numbers into intervals, we can examine whether the function is positive or negative on each interval.

   *The factors with the even powers are always positive. When we reach the points where these factors change sign, the graph will stay on the same side of the x-axis. The factors with odd powers will change sign, the graph will switch sides.

8. Here is a look at the graph. The process was to find the y-intercept, the x-intercepts and the vertical asymptotes. We can then sketch the graph using the analysis of where the graph is positive and negative.

9. Here is a similar example. All that has changed is the powers on two of the factors. The graph will now have a y-intercept of 12, and the same x-intercepts and vertical asymptotes as before, and the same intervals on which it is positive and negative.

   * Let's take a closer look at the end behavior. It is easiest to do so using the expanded form. The numerator has a leading term of x^3 and the denominator has a leading term of x^5. When x is large, the other terms become insignificant, and $f(x)$ will resemble $\frac{x^3}{x^5} = \frac{1}{x^2}$. When x is large, $f(x)$ will tend to zero, so this graph has a horizontal asymptote at zero.

10. Returning to the original example, if we look at $f(x)$ in expanded form, the leading terms are x^5 in the numerator and x^3 in the denominator. When x is large, $f(x)$ looks like x^2, therefore, the end behavior should the same as the end behavior of a parabola. That is, is should go up to the right and up to the left.

11. End behavior of a rational function will follow one of four main behavior patterns. In some cases, it will be easiest to see the behavior of $f(x)$ if we perform long division. If the degree of the numerator is smaller than the degree of the denominator, then when x is large, the denominator will be much larger than the numerator, and the function will tend toward zero. In other words, the graph will tend toward the x-axis as it goes to the far left and the far right. The x-axis is a horizontal asymptote.

    * If the degree of the numerator is the same as the degree of the denominator, then perform long division to yield a constant plus a remainder. The remainder will be small when x is large, so $f(x)$ will tend toward the constant.

    * If the degree of the numerator is one larger than the degree of the denominator, then long division will produce two terms and a remainder. The two terms will resemble to equation of a line. When x is large, the remainder will be insignificant, and $f(x)$ will tend toward the line, which is called an oblique asymptote.

    * Otherwise, the degree of the numerator will be at least two greater that the degree of the denominator. In this case, the end behavior will resemble the graph of a power of x.

12. Here are some simple examples illustrating each case. In the first case, the degree of the denominator is two more than the degree of the numerator, so to the far left and far right, it resembles $y = 1/x^2$

    * In the second case, the numerator and denominator have the same degree, so to the far left and far right, this graph approaches the constant 2.

    * In the third case, to the far left and far right, $f(x)$ approaches the line $3x + 1$

    * Our original example looked like $y = x^2$ to the far left and far right, with some interesting behavior near the y-axis.

13. To recap: To graph a rational function, find the y-intercept. Then find the x-intercepts and vertical asymptotes, which divide the number line into intervals. Analyze the factors to find out on which intervals the function is positive and negative. Perform long division if necessary to analyze the end behavior. Then sketch the graph using this information.