Binomial Theorem

University of Minnesota

Preliminaries and Objectives

Preliminaries

- Pascal's triangle
- Factorials
- Sigma notation
- Expanding binomials

Objectives

- Expand $(x+y)^n$ for $n = 3, 4, 5, \ldots$

Expanding Binomials

$$\begin{aligned}
(x+y)^0 &= 1 \\
(x+y)^1 &= 1x + 1y \\
(x+y)^2 &= 1x^2 + 2xy + 1y^2 \\
(x+y)^3 &= 1x^3 + 3x^2y + 3xy^2 + 1y^3
\end{aligned}$$

Pascal's Triangle

$$\begin{array}{ccccccccccccccc}
&&&&&&&1&&&&&&& \\
&&&&&&1&&1&&&&&& \\
&&&&&1&&2&&1&&&&& \\
&&&&1&&3&&3&&1&&&& \\
&&&1&&4&&6&&4&&1&&& \\
&&1&&5&&10&&10&&5&&1&& \\
&1&&6&&15&&20&&15&&6&&1& \\
1&&7&&21&&35&&35&&21&&7&&1
\end{array}$$

Expanding Binomials

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$

$$(x+y)^4 = (1x^3 + 3x^2y + 3xy^2 + 1y^3)(x+y)$$

$$= 1x^4 + 1x^3y + 3x^3y + 3x^2y^2 + 3x^2y^2 + 3xy^3 + 1xy^3 + 1y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Notation

The notation for the coefficient on $x^{n-k}y^k$ in the expansion of $(x+y)^n$ is

$$\binom{n}{k}$$

It is calculated by the following formula

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

In other words

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Example 1

$$\binom{7}{4} = \frac{7!}{3!4!} = \frac{7x6x5x4x3x2x1}{3x2x1x4x3x2x1} = 35$$

Example 2

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Example 3

$$(x-2)^7 = x^7 + 7x^6(-2) + 21x^5(-2)^2 + 35x^4(-2)^3$$

$$+35x^3(-2)^4 + 21x^2(-2)^5 + 7x(-2)^6 + (-2)^7$$

$$= x^7 - 14x^6 + 84x^5 - 280x^4 + 560x^3 - 672x^2 + 384x - 128$$

Recap

- The expansion of $(x+y)^n$ has terms whose exponents add to n
- The coefficient on $x^k y^{n-k}$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$