1. Linear Growth and Arithmetic Sequences

2. This lesson requires little background material, though it may be helpful to be familiar with representing data and with equations of lines. A brief introduction to sequences of numbers in general may also help.

   In this lesson, we will define arithmetic sequences, both explicitly and recursively, and find both explicit and recursive formulas from the sequence of numbers.

3. Imagine that you begin managing a warehouse that receives boxed shipments. You begin with 37 boxes and each day you receive three more. At the end of Day 1, your warehouse holds 40 boxes. At the end of Day 2, your warehouse holds 43 boxes. This pattern of receiving three boxes per day continues.

   (a) There are several questions that we may be interested in answering:

   - How many boxes will be in the warehouse after Day 9?
   - The capacity of the warehouse is 100 boxes. How long until the warehouse is filled?
   - Can we find a general formula for the number of boxes in the warehouse after Day n?

   (b) To discuss these questions, it is helpful to have a notation for members of the sequence. We will let a_n be the number of boxes in the warehouse after Day n. If we can find an explicit formula for a_n, we can plug values of n into this formula to compute terms of the sequence. For example, if we had the explicit formula, we could plug in 9 to compute the number of boxes after Day 9. From the chart at the left, we can see that $a_1 = 40$, $a_2 = 43$, etc.

   It is easier to find a recursive formula. A recursive formula tells you how to find the next term in the sequence based on the previous term. In this case, the next term is found by taking the previous term and adding 3. A sequence with the recursion of continually adding a constant is called an *arithmetic sequence*. The amount added is called the *common difference* and is denoted d. Thus in general, an arithmetic sequence has a recursion with the next term being the previous term plus the common difference.

   There is only one other piece of information necessary, and that is the starting point. Once we know that $a_1 = 40$, we can find a_2 by adding 3. We continue adding 3 each step to find more terms.

   It doesn't matter where you start, you can find all remaining terms from the recursion. Let's go back to the beginning of the story. You began managing a warehouse that had 37 boxes before the first shipment. Another way to define this sequence is from the starting point of 37 on Day 0.

   This helps us find the explicit formula. To find the number of boxes on Day 9, we merely need to add the shipments we received to the original 37. Since we receive 3 boxes per day, we will have received 9 sets of 3, or 27 boxes by Day 9, giving us a total of 64. In general, we add 3 times the number of days to 37. You many notice that this is the familiar slope-intercept form of a line $y = mx + b$. The input of a linear function is typically x, in sequence notation, the index is typically n. The output of a linear function is typically y, in sequence notation, it is a_n. The y-intercept is where the line starts, so b is similar to the initial term a_0. In the sequence, each term goes up by the common difference d, so d is equivalent to the slope of the line.

   (c) Let's return to the original questions. How many boxes do we have after Day 9? We start at 37, and then add 3 nine times to get 64. Using the recursive formula, this takes quite a bit of calculating. A quicker approach is to use the explicit formula. By plugging in 9 for n, we get the same answer of 64.

   (d) After what day will we have 100 boxes? Now we know the term we are looking for, and want to find out the value of n. We can solve this for n.

4. We are now ready to tackle the four types of problems stated in the introduction. First, given a recursive definition, find the sequence. This is straightforward, given the initial term, merely add the common difference to get successive terms. In this case, we begin with 9, and then add 2 to get each successive term.

5. Next, given an explicit definition, find the sequence. Again, this is straightforward. Plug in the values $1, 2, 3, \ldots$ for n to calculate each term directly. When $n = 1$, we get the first term of 8. When $n = 2$, we get the second term of 3 etc.

6. (a) Given an arithmetic sequence, find the recursive definition. Recall that a recursive definition has two parts, a starting value, and a recursion. We begin by listing that the first term is 54.

   (b) To find the common difference, we can subtract any two consecutive terms. From any term to the next, we are going down by 3, so the next term is the previous term minus 3.

7. We could also begin the sequence at term zero, which in this case would be 57.

8. Let's also find the explicit definition for this sequence. The key is to find a_0, the amount you had before taking the first step. We now get a_n by adding n times the common difference to the starting value. You can verify that the formula is correct by plugging in values for n to see if they match.

9. (a) As a final example, let's try to discover everything about an arithmetic sequence from just two values. We have two terms which are five steps apart. In that space, the values have grown by 30, so each step is of size 6.

   (b) The calculation we just did is essentially the slope formula for lines. We could now begin filling in the values to find the starting value.

   (c) We could also find a_0 by going back seven steps of size 6 from a_7.

(d) To find the recursive definition, list a starting value. For example, we can state that $a_1 = 16$. We could have given the value of a_0 instead. We then need to state the recursion, which in this case is to find the next term by taking the previous term and adding 6.

For the explicit formula, we find the starting point a_0 which is 10, and add to 10 the number n of steps of size 6.

10. To recap: In the recursive definition, we need to define an initial term, typically a_1. We then need to define how to get the next term from the previous term. In an arithmetic sequence, the pattern is to always add or subtract the common difference. In the explicit formula, we need to find the starting location a_0, and then add the common difference n times to arrive at the n^{th} term.