Completing the Square

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Preliminaries and Objectives

Preliminaries
- Expanding binomials like $(x + a)^2$
- General form of a circle, ellipse, parabola, hyperbola

Objectives
- Complete the square

Standard Form of Conic Sections

Standard form of a circle: $\qquad\qquad (x - h)^2 + (y - k)^2 = r^2$

Standard form of a vertical parabola: $\quad y - k = \pm a(x - h)^2$

Standard form of an horiz. parabola: $\quad x - h = \pm a(y - k)^2$

Standard form of an ellipse: $\qquad\quad \dfrac{(x - h)^2}{a^2} + \dfrac{(y - k)^2}{b^2} = 1$

Standard form of a horiz. hyperbola: $\quad \dfrac{(x - h)^2}{a^2} - \dfrac{(y - k)^2}{b^2} = 1$

Standard form of a vertical hyperbola: $\quad \dfrac{(y - k)^2}{b^2} - \dfrac{(x - h)^2}{a^2} = 1$

Example 1

Step 5: Write in standard form

$$y^2 + 4x = 6y - x^2 + 23$$

$$x^2 + 4x + 4 + y^2 - 6y + 9 = 23 + 4 + 9$$

$$(x + 2)(x + 2) + (y - 3)(y - 3) = 23 + 13 = 36$$

$$(x + 2)^2 + (y - 3)^2 = 6^2$$

$$\text{Goal}: (x - h)^2 + (y - k)^2 = r^2$$

This is a circle with radius $= 6$, with center at $(-2, 3)$

Example 2

Step 5: Write in standard form

$$y + x^2 = 8x - 19$$

$$y + 19 + -16 = -x^2 + 8x + -16$$

$$y + 19 + -16 = -(x^2 - 8x + 16)$$

$$y + 19 + -16 = -(x - 4)(x - 4)$$

$$y + 3 = -(x - 4)^2$$

$$\text{Goal}: y - k = A(x - h)^2$$

This is a parabola, pointed downward, with vertex at $(4, -3)$

Example 3

Step 5: Write in standard form

$$4y^2 + 12x = 9x^2 + 4y + 39$$

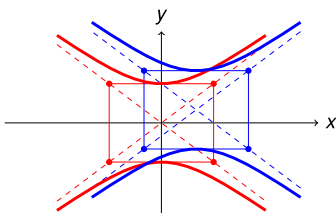$$4y^2 - 4y + 1 - 9x^2 + 12x + -4 = 39 + 1 - 4$$

$$4(y^2 - y + \tfrac{1}{4}) - 9(x^2 - \tfrac{4}{3}x + \tfrac{4}{9}) = 39 + -3$$

$$4(y - \tfrac{1}{2})(y - \tfrac{1}{2}) - 9(x - \tfrac{2}{3})(x - \tfrac{2}{3}) = 39 + -3 = 36$$

$$\frac{(y - \tfrac{1}{2})^2}{9} - \frac{(x - \tfrac{2}{3})^2}{4} = 1$$

Begin with an hyperbola with the y-axis as the transverse axis. The asymptotes have slope $\pm\tfrac{3}{2}$. The vertices are at $(0, 3)$ and $(0, -3)$. This hyperbola is then shifted to the right $\tfrac{2}{3}$ and up $\tfrac{1}{2}$

Example 3

$$\frac{y^2}{9} - \frac{x^2}{4} = 1 \qquad \frac{(y - \tfrac{1}{2})^2}{9} - \frac{(x - \tfrac{2}{3})^2}{4} = 1$$

Recap

Completing the Square:
- Identify the general form and rearrange terms
- Factor out coefficients on x^2 and y^2
- Determine perfect square
- Supply missing constants
- Write equation in standard form