Algebra
Activity 6b - Continuously Compounded Interest

The goal of questions 1 and 2 of this activity is to connect the recursive process for calculating interest to the explicit formula for compounding interest over several time intervals.

1. (a) If you invest \$100 at 5% interest for 1 year, compounded annually, how much money will be in your account at the end of the investment period?
   (b) If you invest \$105 at 5% interest for 1 year, compounded annually, how much money will be in your account at the end of the investment period?
   (c) If you invest \$100 at 5% interest for 2 years, compounded annually, how much money will be in your account at the end of the investment period?
   (d) If you invest \$100 at 5% interest for 3 years, compounded annually, how much money will be in your account at the end of the investment period?
   (e) If you invest \$100 at 5% interest for t years, compounded annually, how much money will be in your account at the end of the investment period?
   (f) If you invest \$100 at 3% interest for t years, compounded annually, how much money will be in your account at the end of the investment period?
   (g) If you invest \$100 at r% interest for t years, compounded annually, how much money will be in your account at the end of the investment period?
   (h) If you invest $\$P$ at r% interest for t years, compounded annually, how much money will be in your account at the end of the investment period?

2. (a) If you invest \$100 at 5% interest for 1 year, compounded twice per year, how much money will be in your account at the end of the investment period?
   (b) If you invest \$100 at 5% interest for 2 years, compounded twice per year, how much money will be in your account at the end of the investment period?
   (c) If you invest \$100 at 5% interest for 5 years, compounded twice per year, how much money will be in your account at the end of the investment period?
   (d) If you invest \$100 at 5% interest for 5 years, compounded quarterly, how much money will be in your account at the end of the investment period?
   (e) If you invest \$100 at 5% interest for 5 years, compounded monthly, how much money will be in your account at the end of the investment period?
   (f) If you invest \$100 at 5% interest for 5 years, compounded n times per year, how much money will be in your account at the end of the investment period?
   (g) If you invest \$100 at 5% interest for t years, compounded n times per year, how much money will be in your account at the end of the investment period?
   (h) If you invest \$100 at r% interest for t years, compounded n times per year, how much money will be in your account at the end of the investment period?
   (i) If you invest $\$P$ at r% interest for t years, compounded n times per year, how much money will be in your account at the end of the investment period?

3. Now let's look at what happens with a fixed investment, a fixed interest rate, a fixed time period, but varying methods of compounding. How much money can we make merely by having the bank calculate interest more frequently?

   (a) If you invest \$100 at 5% interest for 1 year, compounded annually, how much money will be in your account at the end of the investment period?

   (b) If you invest \$100 at 5% interest for 1 year, compounded twice per year, how much money will be in your account at the end of the investment period?

   (c) If you invest \$100 at 5% interest for 1 year, compounded quarterly, how much money will be in your account at the end of the investment period?

   (d) If you invest \$100 at 5% interest for 1 year, compounded monthly, how much money will be in your account at the end of the investment period?

   (e) If you invest \$100 at 5% interest for 1 year, compounded daily, how much money will be in your account at the end of the investment period?

4. Hey, if we can get more money by having our bank compound interest more frequently, why not have them compound the interest every day? every minute? every second? Can we keep getting more money from our bank, or does this process eventually reach a limit?

   Let's examine the expression

   $$\left(1 + \frac{1}{x}\right)^x$$

   When $x = 1$, the expression evaluates to 2. When x gets bigger, the exponent gets bigger (this is good, big exponents = lots of money), but the fraction inside the parentheses gets smaller (this is bad, smaller numbers being multiplied means less growth).

   Evaluate $\left(1 + \frac{1}{x}\right)^x$ when

   - $x = 2$
   - $x = 4$
   - $x = 10$
   - $x = 100$
   - $x = 1,000,000$

   How high can this number get? At what point does it seem to not go any higher?

   (The concept of a limit will be discussed in Calculus. It turns out that this process does have a limit called $e \approx 2.718281828459045$. The proof of this is difficult, it is discussed in courses on Real Analysis.)

5. If you believe that the limit of $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \approx 2.71828$, then we can get an expression for $A(t)$, the amount of money in our account at time t, if $\$P$ is invested at $r\%$ interest compounded continuously (that is, infinitely often)

   - Begin with the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$
   - We need to get the expression inside the parentheses to be $\left(1+\frac{1}{x}\right)$ for some x, so divide both the numerator and denominator of the fraction $\frac{r}{n}$ by r
   - We now need the exponent outside the parentheses to be n/r times something, so multiply and divide the exponent by r.
   - Using the rule $b^{pq} = (b^p)^q$, separate the exponential expression, so that it contains $\left(1+\frac{1}{x}\right)^x$, where $x = n/r$.
   - Simplify the expression using $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

   This is our model for exponential growth. The amount present at time t (in this case, money), $A(t) = Pe^{rt}$, where P is the initial amount, r is the annual rate of growth, and t is time (in years).