1. Completing the Square

2. You should be familiar with expanding binomials, especially perfect squares. This video will talk about completing the square in the context of equations of conic sections like circles, ellipses, parabolas and hyperbolas, so familiarity with those topics may be helpful, although you can learn the competing the square technique without knowledge of conic sections.

   In this lesson, we will develop the technique of completing the square to help us analyze conic sections.

3. The following are the standard forms of the conic sections. Note that any time a squared term occurs, it is simply the variable x or y shifted. There are no extra linear terms. It is easy to find important properties of parabolas, ellipses and hyperbolas, like vertices and centers from these equations.

4. (a) Unfortunately, we may be given an equation in a less desirable form. In order to analyze the shape, we will want to rewrite the equation in standard form. We will use a five step process called completing the square, The idea is to supply missing constants that allow us to factor what remains as a perfect square. Here is an example: The first step is to rearrange the terms to group the x-terms and the y-terms.

   (b) We move the x-terms to the left, and then the y-terms. Since we will want to pick specific constants to add to complete the square, there is a space left for them on the left hand side. We keep the constant 23 on the right side, away from where we will be supplying other constants.

   (c) We next would like to simplify our future multiplication by factoring out the stretching constants. In the case of the circle, we dont have to worry about numbers multiplying x^2 or y^2 since the stretching is taken care of with the value of r, but for other graphs, this will be an important simplification.

   We now wish to figure out how $x^2 + 4x$ might occur in a perfect square, and also how $y^2 - 6y$ might occur in a perfect square.

   (d) We can get $x^2 + 4x$ plus some constant by squaring $x + 2$. Similarly, we can get $y^2 - 6y$ plus some constant by squaring $y - 3$. In the third line, we are looking ahead to what we want to factor, but we dont have the correct constants on the second line. So we need to go back and complete the square by filling in these missing constants.

   (e) When we square $x + 2$, we get $x^2 + 4x + 4$, so we need insert a 4 into the second line. Similarly, when we square $y - 3$, we get $y^2 - 6y + 9$, so we need insert a 9 into the second line to complete the square.

   (f) But now we have changed the equation, we also need to insert these constants on the right hand side. We can then add the constants on the right hand side to get 36.

   (g) To finish the problem, we write the equation in standard form.

   (h) This is the equation of a circle with radius 6, with center at $(-2, 3)$.

5. (a) Here's a parabola. Again, it is important to know where you are headed. We want an equation with the y on the left, and the x-terms on the right. We begin by rearranging the terms.

(b) Since the completing the square will happen on the right side, we move the constant 19 to the left to keep it out of the way. It is hard to factor $-x^2$ into a perfect square, so we factor out the constant multiplying the x-squared.

(c) When we factor out the -1, note that we also factor a -1 from $8x$, making it $-8x$. We are now ready to complete the square. We need to figure out what square will FOIL to $x^2 - 8x$ plus something. We need to cut the $8x$ into two equal pieces of $4x$.

(d) We will FOIL $(x-4)^2$. On the first three lines, we havent changed the equation, we have just moved some terms around. On the fourth line, we have produced a new constant, which we now need to supply to the rest of the lines above.

(e) When we FOIL $(x-4)^2$, we get $x^2 - 8x + 16$, so we supply the 16 on the third line.

(f) But the 16 is inside the parentheses, so we have to distribute the negative sign. On the second line, we need to supply a -16.

(g) We then also supply this -16 to the left side.

(h) We can now write the equation in standard form.

(i) This is a parabola, opening downward, with vertex at $(4, -3)$.

6. (a) Here is a final example involving fractions. First, we rearrange the terms to match our goal.

   (b) Then factor out the coefficients on the squared terms to make completing the square easier.

   (c) We then figure out what terms will be squared. To get $y^2 - y$, we cut the $-1y$ in half, so we will square $y - \frac{1}{2}$.

   (d) Similarly, half of $\frac{4}{3}$ is $\frac{2}{3}$, so we square $(x - \frac{2}{3})$.

   (e) We then multiply to supply the missing constants $\frac{1}{4}$ and $\frac{4}{9}$

   (f) Which are then multiplied by the leading coefficients, giving us 1 and negative 4 to be added to the left side

   (g) We can then write the equation in standard form.

   (h) .

   (i) This is a shifted hyperbola. Start with a hyperbola with the transverse axis on the y-axis. The asymptote box goes up 3 and over 2, so the asymptotes have slope$\pm\frac{3}{2}$. Since the y-term is positive, the vertices are on the y-axis, at $(0, 3)$ and $(0, -3)$. We can then shift this hyperbola to the right $\frac{2}{3}$ and up $\frac{1}{2}$.

   (j) .

7. To recap: Begin by establishing the goal, the general form the answer should be in and rearrange the terms to match the general form. Then factor out the stretch factors, the coefficients on x^2 and y^2. Next, determine the perfect squares to match the general form. Work backward to supply the missing constants and write the equation in standard form.