1. Solving Triangles Using the Law of Cosines

2. You should be familiar with the cosine and inverse cosine functions and the Law of Sines. It may also be helpful to be familiar with geometric proofs of congruent triangles.

   In this lesson, we will use the Law of Cosine to find the missing parts of a triangle in two cases: First, when given the lengths of two sides and the measure of the angle between them. Second, when given the lengths of all three sides.

3. (a) Think of the Law of Cosines as a New and Improved Pythagorean Theorem. It looks much like the Pythagoream Theorem, with an adjustment involving the cosine of the included angle.

   (b) When angle $C = 90°$, the cosine value is 0, so in this case, the Law of Cosines is the Pythagorean Theorem.

4. The Law of Cosines can be expressed using any of the three angles. Note that the side that is isolated on the left is the same as the angle on the right.

5. (a) Given two sides and the included angle of ANY triangle, you can find the third side using the Law of Cosines. Here's an example. Side c is found by using the Law of Cosines.

   (b) You can then find the missing angles using the Law of Sines. Notice that there is only one solution listed. Geometrically, the reason is that Side-Angle-Side is a method for proving congruence of triangles, so there can only be one answer. But why this answer? When finding $\sin^{-1} 0.8988$, only the angle $64°$ is listed, why not the second quadrant angle $180° - 64° = 116°$? The reason is that the longest side of a triangle is opposite the largest angle. Side b is longer than side a. If angle A is bigger than $90°$, and angle B is the biggest angle, then angle B is also bigger than $90°$ and together, angle A and angle B would add to more than $180°$. Therefore, if you use the smaller of the two remaining sides, the angle must be less than $90°$. This is the best approach when using the Law of Sines. Find the smallest angles first. Only the largest angle can be bigger than $90°$.

6. (a) Given all three sides of a triangle, you can find any angle using the Law of Cosines. You will reach a step where you know the cosine value and need to find the angle, so you will need to use the inverse cosine function. Note that it is possible for the Law of Cosines to give a negative value for the cosine, thereby giving an angle over $90°$ from the inverse cosine. You don't need to worry about whether you are looking for a first or second quadrant angle, you will automatically get the one and only answer from the Law of Cosines.

   (b) We can then complete the problem using the Law of Sines. Notice that a is not the longest side of the triangle, and therefore angle A is not the largest angle, and therefore less than $90°$. Also note that originally, we found the angle across from the longest side. This ensures that if there is an angle greater than $90°$, we will find it using the Law of Cosines.

7. Sometimes, you cannot find an angle with the given cosine value. What went wrong? The distance from A to C is 8. If we take any other path from A to C that is not a straight line,

it must be longer than 8. But the remaining two sides have a total length less than 8, which is too short to reach.

8. To recap. Given two sides and the included angle, use the Law of Cosines to find the third side. Then use the Law of Sines to find a second angle. Do not try to find the angle across from the longest side. Then find the third angle by adding to 180°. Given all three sides of a triangle, find an angle by the Law of Cosines. It is best to find the angle across from the longest side. You can then find a second angle by the Law of Sines. To solve Angle-Side-Angle triangles, Angle-Angle-Side triangles, and Side-Side-Angle Triangles, use the Law of Sines.