1. Phase Shift

2. You should be familiar with the graphs of $y = \sin x$ and $y = \cos x$, and how they can be stretched to change the amplitude and period.

   In this lesson, we will shift the wave graphs, both vertically and horizontally. The horizontal shift is called phase shift.

3. (a) In this example the function $2\sin x + 3$ is the sine wave with amplitude 2, shifted up 3. We begin with the standard sine wave.

   (b) We then make the amplitude 2.

   (c) And finally, the plus 3 will move the graph up 3. The tops of the waves move from 2 up to 5, and the bottoms of the waves move from negative 2, up 3 to 1. The centerline of the wave is at 3. The key points on the final graph have the standard x-values of 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π, and y-values 3,5,3,1,3 respectively.

4. The other addition we need to be concerned with is the constant C added inside the parentheses to the angle. Constants added inside the parentheses will shift the graph left and right. For sine and cosine waves, this shift is called the 'phase shift'.

5. (a) In this example, we have a sine wave whose frequency is four times that of a standard sine wave, and the $\frac{\pi}{4}$ will shift the graph horizontally. Again, we begin with a standard sine wave.

   (b) We can then adjust the period. The period is $\frac{2\pi}{B}$, which in this case is $\frac{2\pi}{4}$ or $\frac{\pi}{2}$.

   (c) The red graph goes through a full wave every $\frac{\pi}{2}$, and makes four full waves in reaching 2π.

   The $\frac{\pi}{4}$ will shift the graph horizontally. Again, this shift is called the phase shift. One way to find the phase shift is to find the value for x that makes the angle 0. In this case, the angle is $4x + \frac{\pi}{4}$.

6. We set the angle equal to zero and solve for x. In this case, the phase shift is $-\frac{\pi}{16}$. This will cause us to shift the graph $\frac{\pi}{16}$ to the left.

7. (a) Returning to the unshifted graph. The period is $\frac{\pi}{2}$.

   (b) To mark the standard points, we next find the quarter marks by dividing the period by 4.

   (c) The quarter marks happen every $\frac{\pi}{8}$. The sine wave is at the top at $\frac{\pi}{8}$, back to the middle at $\frac{2\pi}{8}$ or $\frac{\pi}{4}$, at the bottom at $\frac{3\pi}{8}$, and back to the middle at $\frac{4\pi}{8}$, which is $\frac{\pi}{2}$.

   (d) Recall that we found the phase shift to be $-\frac{\pi}{16}$.

   (e) To help mark the x-axis, it is convenient to find a common denominator between the quarter marks and the phase shift. In this case, the common denominator is 16. We can then mark the x-axis every $\frac{\pi}{16}$. Lets restate the key points on the sine wave using the denominator 16. The wave hits the high point at $\frac{2\pi}{16}$, back to the middle at $\frac{4\pi}{16}$, down to the bottom at $\frac{6\pi}{16}$ and back to the middle at $\frac{8\pi}{16}$.

   (f) We now need to shift the wave $\frac{\pi}{16}$ to the left. The start of the wave moves to $-\frac{\pi}{16}$, then to the top at $\frac{\pi}{16}$, back to the middle at $\frac{3\pi}{16}$, down to the bottom at $\frac{5\pi}{16}$, and back to the middle at $\frac{7\pi}{16}$. We can extend the wave further if we wish.

8. (a) Let's review the process for drawing sine and cosine waves. First, mark the y-axis as follows: D determines the vertical shift, which is where we place the centerline. From the centerline, go up and down according to the amplitude.
   (b) Then we mark the x-axis. First, find the period, $\frac{2\pi}{B}$, divide by four to get the quarter marks, and find the phase shift by setting the angle to zero and solving for x. Next, find a common denominator between the quarter marks and the phase shift and mark the x-axis evenly. You can then mark the reference points for the unshifted graph, and then use the phase shift to move the graph left or right.

9. (a) Here's a full example. It is a cosine wave, so our instinct is to follow the pattern top, middle, bottom, middle, top. Except this cosine wave has a negative sign in front, which flips the graph top to bottom, so it will go bottom, middle, top, middle, bottom. It has an amplitude of 6.
   (b) A vertical shift of 2
   (c) So the centerline will be at 2, the top will be 6 above 2 which is 8, and the bottom will be 6 below 2 which is -4.
   (d) The period is $\frac{2\pi}{B}$, which in this case is $\frac{2\pi}{2}$, which is π.
   (e) The quarter marks happen every $\frac{\pi}{4}$.
   (f) To find the phase shift, we set the angle, $2x - \frac{\pi}{3} = 0$ and solve for x. The phase shift is $\frac{\pi}{6}$
   (g) We find a common denominator between $\frac{\pi}{4}$ and $\frac{\pi}{6}$, which is 12, and we rewrite the quarter marks and the phase shift in terms of $\frac{\pi}{12}$. When we mark the x-axis, we will mark it every $\frac{\pi}{12}$. You may wish to jot down this information to refer to as we move ahead to drawing the graph.

10. (a) Let's mark the y-axis first. The centerline is at 2, the bottom at negative 4 and the top at 8.
    (b) We mark the x-axis every $\frac{\pi}{12}$, emphasizing the quarter marks at $\frac{\pi}{4}$ (which is $\frac{3\pi}{12}$), $\frac{\pi}{2}$ (which is $\frac{6\pi}{12}$), $\frac{3\pi}{4}$ (which is $\frac{9\pi}{12}$) and π (which is $\frac{12\pi}{12}$).
    (c) The reference points for a negative cosine wave are placed at the bottom, middle, top, middle and bottom.
    (d) We can then draw the unshifted graph, extending the wave as far as we wish by following the pattern.
    (e) Finally, we shift the reference points by the phase shift, which was $\frac{\pi}{6}$, or $\frac{2\pi}{12}$. We can then draw the final graph.

11. (a) Now let's go backward. Given the graph, can we find an equation?
    (b) First we need to decide on the general form. We have several choices. Perhaps we can choose a section that looks like a standard cosine wave.
    (c) The middle points are at a height of -1, so D is -1
    (d) The graph hits it peak at 1 and its valley at -3, which is a total distance of 4. The amplitude is half the distance from the bottom to the top, which in this case is 2.
    (e) B is determined by the period. One full wave begins at $\frac{\pi}{12}$ and ends at $\frac{9\pi}{12}$, for a wavelength of $\frac{8\pi}{12}$. $\frac{8\pi}{12}$ is $\frac{2\pi}{B}$, so $B = 3$.

(f) Finally, we know the phase shift is $\frac{\pi}{12}$. The phase shift is the value of x that makes the angle 0. In this case, the angle is $3x+C$, so we have the equation, $3x+C=0$, and know that the x-value should be $\frac{\pi}{12}$, so we solve the equation $3\left(\frac{2\pi}{12}\right)+C=0$ giving $C=-\frac{\pi}{4}$. This is one possible way to write the equation for the graph.

12. (a) Let's do this again, this time focusing on a portion of the graph that looks like a standard sine wave.

    The analysis of the amplitude, vertical shift and period are the same as before, the amplitude is 2, the centerline is at -1 and the period is $\frac{8\pi}{12}$ so $B=3$.

    (b) This time, the phase shift is $-\frac{\pi}{12}$, so $-\frac{\pi}{12}$ is the value for x which makes the angle 0. We need to solve $3\left(-\frac{\pi}{12}\right)+C=0$, which gives $C=\frac{\pi}{4}$.

    (c) .

13. (a) Let's try it one more time, with an upside-down cosine wave.

    Again, the amplitude is 2, the centerline is at -1 and $B=3$.

    (b) The phase shift is $-\frac{3\pi}{12}$, so $3\left(-\frac{3\pi}{12}\right)+C=0$, making $C=\frac{3\pi}{4}$.

    (c) This gives us the equation of the graph. You can find many more equations for this graph. All you need to do is find one complete wave, and figure out the values of A,B,C and D from that wave.

14. To recap: D determines the centerline of the wave. The amplitude is then used to find the tops and bottoms of the waves. The period is $\frac{2\pi}{B}$, which is cut into four parts to determine the reference points. To find the phase shift, find the value of x that makes the angle 0.