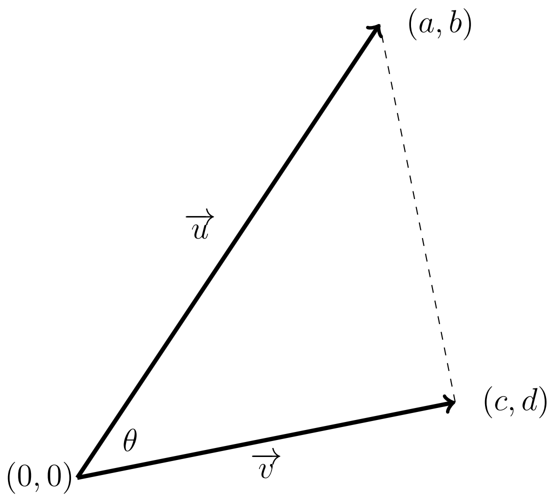
The goal of this worksheet is to discover a simple formula for the area of a triangle when two sides are given as vectors. Those well-versed in Linear Algebra will recognize this as the determinant (almost, the determinant is the area of the parallelogram). It is not necessary to discuss the Linear Algebra with the students, but the fact that such a simple formula exists is noteworthy.

Suppose we have a triangle with one vertex at the origin. We can express two sides of the triangle as vectors $||\overrightarrow{u}|| = \langle a, b\rangle$ and $||\overrightarrow{v}|| = \langle c, d\rangle$

We wish to write the area of the triangle in terms of $a, b, c, d$




1. Write the area of the triangle using $\sin\theta$, $||\overrightarrow{u}||$ and $||\overrightarrow{v}||$

   This is a reminder of a formula for the area of a triangle which the students found in Activity 4b. They will need to sort out the notation, be careful with the difference between a vector and the length of a vector. $\boxed{\text{Area} = \frac{1}{2}||\overrightarrow{u}|| \cdot ||\overrightarrow{u}|| \cdot \sin\theta}$

2. Find the direction of $\overrightarrow{u}$. That is, find $\theta_u$ in terms of $a$ and $b$

   This is a review of the Cartesian to polar conversions. $\boxed{\theta_u = \arctan\frac{b}{a}}$.

   You can assume everything is in the first quadrant for the time being and ignore the fact that the arctan will not be correct if $a$ is negative. You may wish to come back to the issues involving the quadrants at the end.

3. Find the direction of $\overrightarrow{v}$. That is, find $\theta_v$ in terms of $c$ and $d$ $\boxed{\theta_v = \arctan\frac{d}{c}}$

4. Express $\theta$ in terms of $\theta_u$ and $\theta_v$ $\boxed{\theta = \theta_u - \theta_v}$

5. Express $\sin\theta$ using inverse trig functions and $a, b, c, d$

   Combined with the next step below

6. Use the angle sum/difference formula to write $\sin\theta$ in terms of $a, b, c, d$

$$\begin{aligned}
\sin\theta &= \sin\left(\theta_u - \theta_v\right) \\
&= \sin\left(\arctan\frac{b}{a} - \arctan\frac{d}{c}\right) \\
&= \sin\left(\arctan\frac{b}{a}\right)\cos\left(\arctan\frac{d}{c}\right) - \cos\left(\arctan\frac{b}{a}\right)\sin\left(\arctan\frac{d}{c}\right) \\
&\text{Draw triangles here} \\
&= \frac{b}{||\overrightarrow{u}||}\cdot\frac{c}{||\overrightarrow{v}||} - \frac{a}{||\overrightarrow{u}||}\cdot\frac{d}{||\overrightarrow{v}||} \\
&\text{Students may also use } \sqrt{a^2+b^2} \text{ etc. in the denominator}
\end{aligned}$$

7. Write the area of the triangle in terms of $a, b, c, d$

   $$\text{Area} = \frac{1}{2}||\overrightarrow{u}||\cdot||\overrightarrow{u}||\cdot\sin\theta = \frac{1}{2}||\overrightarrow{u}||\cdot||\overrightarrow{u}||\cdot\left(\frac{b}{||\overrightarrow{u}||}\cdot\frac{c}{||\overrightarrow{v}||} - \frac{a}{||\overrightarrow{u}||}\cdot\frac{d}{||\overrightarrow{v}||}\right) = \boxed{\tfrac{1}{2}\left(bc - ad\right)}$$

8. If $\overrightarrow{u} = (5, 1)$ and $\overrightarrow{v} = (4, -2)$, find the area.

   Students will likely just plug the numbers in without thinking about the quadrant. This is fine. At the end, you may wish to deal with the quadrant issues. $\boxed{\text{Area} = \frac{1}{2}(4 - (-10)) = 7}$

9. If $\overrightarrow{u} = (4, -2)$ and $\overrightarrow{v} = (5, 1)$, find the area. Do you need to adjust the formula?

   $$\boxed{\text{Area} = \frac{1}{2}(-10 - 4) = -7}$$

   Students may make some comments about the fact that area can't be negative, which is the main point. They may say to take the absolute value. There are connections to physics that give the negative sign meaning. Those well-versed in Linear Algebra will notice the orientation (clockwise vs. counterclockwise).