Trigonometry
Activity 1b - Piston Motion Instructor's Guide

The goal of this worksheet is to derive a formula that gives the location of the end of a rod whose motion is being controlled by a piston. While this task seems daunting at first, it can be broken into small steps. In the past, students have found problems 9-11 the most difficult, in which they have to generalize a process. It requires some familiarity with this abstraction process, and does not need to be done in Unit 1. We often wait until after Unit 2 on graphing before doing this activity

Questions 1-4 is largely an exercise on recognizing point A is moving around the unit circle. A first step toward abstraction is getting students to realize in question 4 that they don't have a good way to express the coordinates other than to use the cosine and sine functions. Use of the desmos demonstration will help students visualize, or you may wish to have students diagram the positions. A common mistake I've seen in the past is for students to omit the parentheses from coordinates, e.g., x, y instead of (x, y). Be on the lookout for this!

1. If $\theta = 90°$, the coordinates of A are $\boxed{(0, 1)}$.

2. If $\theta = 60°$, the coordinates of A are $\boxed{(\frac{1}{2}, \frac{\sqrt{3}}{2})}$.

3. If $\theta = 30°$, the coordinates of A are $\boxed{(\frac{\sqrt{3}}{2}, \frac{1}{2})}$.

4. If $\theta = 17°$, the coordinates of A are $\boxed{(\cos 17°, \sin 17°)}$ Decimal answers are OK here, but have students describe how they found the answer.

   Questions 5-8 involve dropping an altitude from the point A (as we did in the Special Triangles worksheet), and dividing the problem into two right triangles. Discourage students from using decimals here, except perhaps in question 8.

5. If $\theta = 90°$, the coordinates of B are $\boxed{(2\sqrt{2}, 0)}$.

6. If $\theta = 60°$, the coordinates of B are $\boxed{(\frac{1+\sqrt{33}}{2}, 0)}$.

7. If $\theta = 90°$, the coordinates of B are $\boxed{(\frac{\sqrt{3}+\sqrt{35}}{2}, 0)}$.

8. If $\theta = 17°$, the coordinates of B are $\boxed{(3.942, 0)}$.

   The general process for questions 5-8 is to drop an altitude from A, find the width of the base of each triangle, and add them together. It is best to have students diagram this process, getting different students to draw different problems.

9. Describe in words the step-by-step process you are using to find the coordinates of B given the angle θ.

   If students are stuck on question 8, it might be good to move ahead to this problem. Developing an algorithm may be new to students, so getting them to think about the process is crucial.

10. For an arbitrary angle θ, the coordinates of A are $\boxed{(\cos(\theta), \sin(\theta))}$.

    Refer students back to question 4 if they are stuck. This is a good chance to talk about what we mean when we say 'arbitrary'.

11. For an arbitrary angle θ, the coordinates of B are $\boxed{(\cos(\theta) + \sqrt{9 - \sin^2(\theta)}, 0)}$.

    Here, students can be encouraged to follow the procedure in question 9. This is a good opportunity to have the students who were last to figure out the process explain the steps.

12. The previous answer expresses the x-coordinate of B as a function of θ. Graph this function. Describe what the graph looks like. The biggest obstacle here is reorienting oneself from coordinates of the form $(x, f(x))$ or (x, y) to $(\theta, \cos(\theta) + \sqrt{9 - \sin^2(\theta)})$. The independent variable doesn't always have to be x. The graph looks like a sine wave, almost. There is a flattening at the bottom of the waves.

13. When the length of the second rod is 10, the coordinates of A are unchanged and the coordinates of B are $\boxed{(\cos(\theta) + \sqrt{100 - \sin^2(\theta)}, 0)}$.

    Also graph this function.

    This graph looks more like a sine wave with midline at 9 and amplitude 1. The disturbance on the bottom is less noticeable.

14. When the length of the second rod is 1, the coordinates of A are unchanged and the coordinates of B are $\boxed{(\cos(\theta) + \sqrt{1 - \sin^2(\theta)}, 0)}$

    Also graph this function.

    This graph looks odd because B slides back and forth along the *positive* x-axis only when A is on the right half of the unit circle. The piston gets stuck when A is on the left half of the unit circle. This may be physically impossible, the black to blue connecting rod is no longer pulling the length 3 rod. When the length of the second rod is 1, and when $90° \leq \theta \leq 270°$, the x-coordinate of A is ≤ 0, so the farthest right that B can slide is the origin. There are many, many different ways to explain this concept. Feel free to come up with one better than mine.

15. When the length of the second rod is n, the coordinates of A are unchanged and the coordinates of B are $\boxed{(\cos(\theta) + \sqrt{n^2 - \sin^2(\theta)}, 0)}$.

Some questions you can use to challenge students:

- What if the length of the first rod is n and the length of the second rod is m?
- Look over responses to (9) very carefully. Perhaps have two students compare their answers and see if each could find the coordinates of B based solely on their classmate's description.