The goal of this worksheet is to prepare students for handling inverse trig functions by giving them a sense of what it means to be an inverse function. By the end of this worksheet, students should understand that inverse functions have one output per input, which is to say that the original function has one input associated with a given output. Also, inverse functions might not always compose to the identity. For example, $\sqrt{(-2)^2} \neq -2$. The key issue is the restriction of the domain in the case when the original function is not one-to-one. There is a lot of technical jargon (like functions being 'one-to-one') that is not necessary to get students to understand the issue, which is easiest to understand in terms of the square root function. When inverting $y = x^2$, a choice needs to be made as to which square root, the positive root or the negative root, is chosen to be the inverse. That is the main theme of the activity, you can invert most functions that appear in an algebra course, but square roots have an issue that needs to be resolved. This is precisely the issue that needs to be resolved for arcsin and arccos.

Many functions are used in this activity as a review of algebra. Students may have trouble recalling what a logarithm is, and this is a good time to review the basics.

1. What operation will undo the following?
   (a) adding 3 subtracting 3
   (b) subtracting 8 adding 8
   (c) multiplying by -2 dividing by -2
   (d) dividing by 6 multiplying by 6
   (e) multiplying by 0 A common response you will get to this question is "adding x." E.g., I've seen a student say that if $f(3) = 0 \times 3 = 0$, then the inverse operation is to add 3. It's important to stress here that the output of an inverse function should be defined independent of knowing what value was input into f in the first place. This is actually a very deep question, and students should realize that this function can't be undone. It is the worst possible example of a function not being one-to-one. You don't really need to dwell on this, other than students should realize that this operation can't be undone. They may have some clever responses if you ask them why, though I'm not sure how I would respond.
   (f) taking a reciprocal This is a fun one to let students struggle with! The answer is to take the reciprocal again. Try some examples with fractions, and then with whole numbers (because the reciprocal of 2 is harded to grasp than the reciprocal of, say, $\frac{3}{4}$)
   (g) squaring Taking a square root. Ask students to find the square root of -4. Ask students to square 2 and to square -2. Ask students to find the square root of 4. Why did they choose the answer they chose? This is the main point of this activity, to discuss the issues involved with the square root. At some point, students should graph $y = x^2$. It is good to get students talking early this issue, though it need not be fully developed here, there are more opportunities later to talk about the details.
   (h) cubing Taking the cube root. This is well-defined. You may want to have students graph $y = x^2$ and $y = x^3$. Ask students if they see something different in the graphs. You can also ask students to find the cube root of -8 and the square root of -4 to point out the difference.

(i) raising 10 to a power Taking the $\log_{10}$. This is a good opportunity to review the definition of logarithm. At a basic level, what is a logarithm?

When is it not possible to undo an operation? You may wish to have students draw graphs and see what they notice. The 'divide by zero' is a tricky one to answer. Some students may know how the graph of an inverse relates to the graph of a function, and this may be something for the fastest-working students to explore. This is a good time to try to solidify the understanding of the issue in the square root. There is a separate issue about domain/range for square roots, exponentials and logarithms which can be explored.

2. For each of the above, write a functional equation for the operation and for the inverse.

   Example: Adding 3, the equation is $y = x + 3$. For the inverse, subtracting 3, the equation is $y = x - 3$

   Writing the above operations as equations should be fairly straightforward. You may wish to talk about the location of the minus sign for dividing by -2. There is a standard simplification, but keeping the minus sign in the denominator with the '2' may illustrate the point better. For $y = 0x$, if students instead write $y = 0$ (or simplify $y = 0x$ to $y = 0$) they may have some comments about the x not being present. This is also a good time to talk about $y = \pm sqrtx$ being two answers to the inverse.

3. For each of the above, write the function $f(x)$ and inverse function $f^{-1}(x)$ in function notation.

   Example: Adding 3, the function is $f(x) = x + 3$. For the inverse, subtracting 3, the function is $f^{-1}(x) = x - 3$

   This emphasizes the function notation, and is a good time to review that notation, and the concept of x being the input to a function. Writing the above operations as functions should be fairly straightforward. What students may not immediately realize, however, is what the composition of these functions results in. For those who are working quickly through this section, ask them 'what happens if they compose a function with its inverse function?' Some numerical examples may help.

4. For each inverse equation in question 2), solve the equation for x.

   What is the relationship between the equation of a function and its inverse?

   Students should realize here that the inverse function is the function where x and y have switched roles. (e) will not be possible and (g) can lead to an interesting discussion. 'Does it matter whether you choose the $+$ or $-$ answer when undoing the square root?'

5. For each function $f(x)$ in question 3), find $f(-2)$.

   For each inverse function $f^{-1}(x)$ in question 3), find $f^{-1}(-2)$

   This is a chance to refresh the understanding of function notation. The domain question comes up when trying to find $\sqrt{-2}$. This is a separate issue from the finding the inverse question. This is also a good time to check understanding of negative exponents.

6. Solve each equation for x

   (a) $1 = x + 3$
   (b) $-10 = x - 8$
   (c) $4 = -2x$
   (d) $-\frac{1}{3} = \frac{x}{6}$
   (e) $0 = 0x$
   (f) $-\frac{1}{2} = \frac{1}{x}$
   (g) $4 = x^2$
   (h) $-8 = x^3$
   (i) $0.01 = 10^x$

   How do the steps used to solve these equations relate to the inverse functions in question 2? Which of the equations produced multiple answers?

   Students should comment about how the solving process uses the inverse functions. You are trying to eliminate the other stuff near x by undoing the operations with their inverses. Students should also realize that -2 works in each equation, though the natural answer for the square root is 2. This is an opportunity to talk about the multiplicity of answers. Note that -2 also works in $0 = 0x$.

7. For each of the equations in question 2), graph the function and its inverse.

   What is the relationship between the graph and its inverse?

   This is a good time to use *desmos.com*. Students should see that graphs reflected across the line $y = x$, though graphing tools may need to be squared up in order to see this. Have students comment on all of the different ways that x and y interchange: they switch roles in the equation, domain and range switch, in the reflection, the x and y-axis are switched, certain aspects of specific graphs, like intercepts and/or asymptotes switch.

8. Examining the graphs in question 7), what property of the function is necessary so that the inverse function produces just one answer? What happens when this property is lacking?

   Students should recognize something about the graph $y = x^2$ decreasing, then increasing, which causes the multiple answers for the inverse. This can lead into a discussion of what needs to be done to define the inverses of cosine and sine.