Trigonometry
Activity 2a - Transformations of Functions and their Graphs Instructor's Guide

By the end of this worksheet, you should be able to describe how the graph of

$$y = A \cdot f(Bx + C) + D$$

differs from the graph of $y = f(x)$, for any **arbitrary** function f. Your goal is to figure out how the parameters A, B, C, and D affect the graph. We use some familiar (or not-so-familiar) functions, like parabolas, exponential functions, logarithmic functions, and reciprocals.

The motivation of this worksheet is to prepare students for transforming graphs of sine and cosine waves. (amplitude, frequency, wavelength, phase shift) **We recommend that students bring laptops to class.** The instructor may need to bring adapters. Refer students to *desmos.com*, which also works on mobile devices!

The instructions refer to 'Task' 1, 2, 3. You may wish to have students work in groups of three, with each student doing one of the three tasks, and then comparing results. This activity also works well with nine students, having three students working on one task, and then have the sets of three students compare results in the group of nine.

1. Translations

   - Task 1: Graph $y = x^2, y = x^2 + 4$, and $y = x^2 - 2$
   - Task 2: Graph $y = x, y = x + 3$ and $y = x - 1$
   - Task 3: Graph $y = x^2, y = (x-3)^2$, and $y = (x+1)^2$
   - What do you notice about the shape of the graphs? the location of the graphs?

     It is best if students are allowed to use vocabulary familiar to them. 'Translation' may be too formal of a word to use. Students should notice that graphs maintain their shape, but are moved/shifted/translated/lifted.
   - What do the three tasks have in common? How do they differ?

     If students didn't make this comment in the previous step, they should notice that some graphs were shifted left/right and others up/down. Ask students how the function notation determines the direction.
   - What effect will adding D to a function value have on a graph?

     Shifting the graph *up* by D units, or adding D to each y-coordinate.

     What effect will adding C to the input value (x) before applying the function have on the graph?

     Shifting the graph *left* by C units, or subtracting C from each x-coordinate. Here, it is important to note that the horizontal shift is counterintuitive. Adding C moves the graph **left**, not right, while addind D moved the graph up, as we would expect.

- Interpret $y = x - 1$ and $y = (x - 1)$ in two different ways and show that their graphs will be the same.

  (Method 1) Ask students why *subtracting* 1 from all the y-coordinates in $y = x$ is the same as *adding* 1 to all the x-coordinates. This is the algebraic explanation. You may wish to introduce a slightly tougher notation. Start with $y = f(x) = x$ and ask why $f(x) - 1$ is the same as $f(x - 1)$ in this case. Will this always work?

  (Method 2) Alternatively, based on their answer ot the previous question, why is shifting down by 1 the same as shifting right by 1 in this case? Would this work if the slope were 2 instead of 1? This is a fairly advanced question, it is OK to skip it, but it is a good question to find out who your best visual students are.

- Second task for all groups:

  Graph $y = \dfrac{1}{x}, y = \dfrac{1}{(x-3)}$ and $y = \dfrac{1}{x} + 2$

  Students should predict what will happen to the graph before they graph the second and third functions. You can repeat this with other functions like $y = e^x$ and $y = \log x$.

- If you know what the graph of $y = \sin(x)$ looks like, can you describe what the graphs of $y = \sin(x) + 4$ and $y = \sin(x - \frac{\pi}{4})$ look like?

  These graphs should be shifted up by 4 units and right by $\frac{\pi}{4}$ units, respectively.

  Ask students to identify which operations (out of $\boxed{\text{addition}}$, $\boxed{\text{subtraction}}$, multiplication, and division) correspond to translations. Why do the remaining two operations have a different effect? When is a translation vertical, and when is it horizontal?

2. Reflections

   This is a good time for students to pay attention of how negative $(-)$ signs factor through parentheses. Be on the lookout for students who assert that $-\ln(x) = \ln(-x)$

   - Task 1: Graph $y = e^x, y = e^{-x}, y = -e^x$, and $y = -e^{-x}$
   - Task 2: Graph $y = \ln(x), y = \ln(-x), y = -\ln(x)$ and $y = -\ln(-x)$
   - Task 3: Graph $y = \sqrt{x}, y = \sqrt{-x}, y = -\sqrt{x}$, and $y = -\sqrt{-x}$
   - What effect will placing a negative sign in front of the function value do to the graph? $\boxed{\text{Reflect it vertically over the } x\text{-axis.}}$ What effect will placing a negative sign on the input value before applying the function have on the graph? $\boxed{\text{Reflect it horizontally over the } y\text{-axis.}}$ It is helpful to talk about the placement of the negative sign on x having some common sense connection to the x-direction, for example, the reflection is $\boxed{\text{horizontal}}$
   - Second task for all groups: Graph $y = x^2, y = -x^2, y = (-x)^2$, and $y = -(-x)^2$
   - Why do the graphs of $y = x^2$ and $y = (-x)^2$ look the same? Give two reasons: one by simplifying the second equation algebraically, and one by interpreting the effect of the negative sign on the graph.

     $(-x)^2 = (-x)(-x) = (-1) \cdot x \cdot (-1) \cdot x = (-1)^2 x^2 = 1x^2 = x^2$

     Because the graph of $y = x^2$ is symmetric, reflecting it horizontally does not affect the appearance of the graph.

- If you know what the graph of $y = \sin(x)$ looks like, can you describe what the graph of $y = -\sin(x)$ and $y = \sin(-x)$ look like?

  Ask students what kind of operations correspond to reflecting a graph. Implicitly, we are multiplying by -1. For students who progressing through the worksheet quickly, ask them to provide examples of functions for which $f(x) = -f(-x)$, for example. One such function is $f(x) = x^3$. Can they prove it both algebraically and visually? Start to hint at the discovery of even functions ($f(-x) = f(x)$) and odd functions($f(-x) = -f(x)$).

3. Magnifications

- Task 1: Graph $y = \ln(x), y = 4\ln(x)$, and $y = \ln(4x)$ Make special note of where the graphs cross the x-axis, to be used as a reference point
- Task 2: Graph $y = \sqrt{x}, y = 2\sqrt{x}$ and $y = \sqrt{2x}$
- Task 3: Graph $y = \frac{1}{x}, y = \frac{2}{x}$, and $y = \frac{1}{2x}$

  Emphasize the difference between $y = 1/2x$ and $y = 1/(2x)$ on a calculator/graphing app
- Second task for all groups: Graph $y = x^2, y = 2x^2$, and $y = 5x^2$
- What effect will multiplying A to a function value have on the graph?

  $\boxed{\text{Stretching the graph vertically by a factor of } A}$

  What effect will multiplying B to the input value (x) before applying the function have on the graph?

  $\boxed{\text{Compressing the graph horizontally by a factor of } B}$
- If you know what the graph of $y = \sin(x)$ looks like, can you describe what the graphs of $y = A\sin(x)$ and $y = \sin(Bx)$ look like?

  A will make the graph taller (y-direction), B will make the graph thinner (x-direction, but counterintuitive - thinner, not wider).

4. Summary

   Given the graph of a function $y = f(x)$ and the transformed graph $y = \pm A \cdot f(\pm Bx + C) + D$:

   - Which things in the transformation affect the graph horizontally (left and right)? Numbers applied to the *input* of a function, i.e., the things closest to x

     Which things in the transformation affect the graph vertically (up and down)? Numbers applied to the output of a function, i.e., the things farthest from x

   - How does a multiplier affect the graph? Stretch/compress A minus sign? Reflect A number added? Shift/translate

   - If you know the graph of $y = f(x)$, how can you find the graph of $y = -5f(4x) - 3$? The graph of $y = -5f(4x + 2) - 3$?

     The main issue is the abstractness of the function f. Get students to accept that f determines the shape. f makes the parabola, or wave, or smiley face or whatever. Beginning with that shape, how do we stretch, flip, shift the graph? Paying attention to order of operations is critical. I like to think of points (x, y) in the graph of $f(x)$ being sent to $(\frac{x}{4} - 2, -5y - 3)$. One of the key ideas is to think about the order of operations of the vertical changes. After figuring out what is inside the parentheses, the function is then multiplied by A, then potentially a negative sign on A, then D is added. It is also worth mentioning that subtraction is not treated as something different than addition. Subtraction is merely adding a negative number. Likewise, division is multiplying by the reciprocal. The order in the horizontal direction is trickier, and can be downplayed.