**Trigonometry**
**Activity 3d - Double Angle Formulas** Instructor's Guide

This worksheet develops several more trig formulas. By now, students are typically pretty adept at the algebraic manipulations. As always, pay close attention to the notation the students are using; there typically are things that can be cleaned up with their presentation. The activity is very heavy in algebra and simplification of fractions. We recommend motivating the class with an example such as 'What is $\sin(15°)$?' Ask students how $15°$ can be written as double or half of an angle. Dispel any myths that $\sin(15°) = \sin(30°)/2$.

1. **Alternate Versions of the Pythagorean Theorem**

   It may be helpful to draw a unit circle at a table's whiteboard. What are the $(x, y)$-coordinates of where an arbitrary angle touches the unit circle? Draw the inscribed triangle and label them with the side-lengths.

   $\sin^2\theta + \cos^2\theta = 1$

   It may also be good to ask what $\sin^2(2A) + \cos^2(2A)$ equals?

   Subtract $\cos^2\theta$ from both sides to get a formula for $\sin^2\theta$

   $\sin^2\theta = \boxed{1 - \cos^2\theta}$

   Subtract $\sin^2\theta$ from both sides to get a formula for $\cos^2\theta$

   $\cos^2\theta = \boxed{1 - \sin^2\theta}$

2. **Double Angle Formula for Sine**

   Ask students what is happening in this problem. They should be pretty explicit about how the substitution works.

   $\sin(A + B) = \sin A \cos B + \cos A \sin B$

   $\sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \sin\theta\cos\theta = \boxed{2\sin\theta\cos\theta}$

3. **Double Angle Formula for Cosine**

   Ask students to supply reasons for their steps. They should be improving their presentation, both written and orally.

   $\cos(A + B) = \cos A \cos B - \sin A \sin B$

   $\cos(2\theta) = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \boxed{\cos^2\theta - \sin^2\theta}$

   Find two more formulas for $\cos(2\theta)$ by using the Alternate Versions of the Pythagorean Theorem

   $\cos(2\theta) = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = \boxed{1 - 2\sin^2\theta}$

   $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = \boxed{2\cos^2\theta - 1}$

4. **Double Angle Formula for Tangent**

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(2\theta) = \tan(\theta+\theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \boxed{\frac{2\tan\theta}{1-\tan^2\theta}}$$

5. **Half Angle Formula for Sine**

Some good questions: 'What is the relationship between $A$ and $2A$? What is the relationship between $\theta$ and $\theta/2$? If $2A = \theta$, then $A = ?$'

Begin with $\cos(2A) = 1 - 2\sin^2(A)$. Let $2A = \theta$. Rewrite the equation in terms of $\theta$ and solve for $\sin(\frac{\theta}{2})$.

THIS IS NOT A TYPO. The sine function on the right hand side will be solved for in terms of $\cos\theta$, but this is correct, the left hand side begins $\cos\theta$.

$$\begin{aligned} \cos\theta &= 1 - 2\sin^2\left(\frac{\theta}{2}\right) \\ \implies 2\sin^2\left(\frac{\theta}{2}\right) &= 1 - \cos\theta \\ \implies \sin^2\left(\frac{\theta}{2}\right) &= \frac{1-\cos\theta}{2} \\ \implies \sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \end{aligned}$$

For the more advances students, you may ask how we can determine whether the square root should be + or -

6. **Half Angle Formula for Cosine**

Begin with $\cos(2A) = 2\cos^2(A) - 1$. Let $2A = \theta$. Rewrite the equation in terms of $\theta$ and solve for $\cos(\frac{\theta}{2})$.

$$\begin{aligned} \cos\theta &= 2\cos^2\left(\frac{\theta}{2}\right) - 1 \\ \implies 2\cos^2\left(\frac{\theta}{2}\right) &= 1 + \cos\theta \\ \implies \cos^2\left(\frac{\theta}{2}\right) &= \frac{1+\cos\theta}{2} \\ \implies \cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1+\cos\theta}{2}} \end{aligned}$$

7. **Half Angle Formula for Tangent**

   Simplify as much as possible:

   For most students, the first step is an acceptable answer. More advanced students can be encouraged to simplify, but it is probably best to show them the form that they are tying to simplify toward. You can hint that rationalizing the denominator is the technique.

   For those students going further, remind them of the distributive property for square roots. Once they see this, they should recognize that they're dividing by a fraction so will need to multiply by a reciprocal.

$$\begin{aligned}
\tan\left(\frac{\theta}{2}\right) &= \frac{\sin(\frac{\theta}{2})}{\cos(\frac{\theta}{2})} \\
&= \frac{\sqrt{\frac{1-\cos\theta}{2}}}{\sqrt{\frac{1+\cos\theta}{2}}} \\
&= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \cdot \sqrt{\frac{1+\cos\theta}{1+\cos\theta}} \\
&= \sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}} \\
&= \sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}} \\
&= \frac{\sin\theta}{1+\cos\theta}
\end{aligned}$$

   From here we can carry on...

$$\begin{aligned}
\frac{\sin\theta}{1+\cos\theta} &= \frac{\sin\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{1-\cos\theta} \\
&= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} \\
&= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} \\
&= \frac{1-\cos\theta}{\sin\theta}
\end{aligned}$$

Note that the $\pm$ has been dropped from all expressions. It turns out that the values of $\sin\theta$ and $\cos\theta$ are enough to get the correct value for $\tan\left(\frac{\theta}{2}\right)$, but care should be taken to prove this for the most advanced students.