Trigonometry
Activity 4a - Congruent Triangles Instructor's Guide

In this activity, you will discover the issues involved in using the Law of Sines and Law of Cosines to solve triangles. You will be given three parts of a triangle (side lengths and/or angle measures) and will be asked to place points to produce a triangle that has these three parts. The questions are intentionally open ended, there may be many different ways to produce triangles with the given parts. When working with your group members, you should try to produce as many original answers as possible.

Answers are different if the parts are different. The SSA case is the interesting one.

Two triangles are the same if they are **congruent**. That is, two triangles are the same if you can move one triangle to line up with another by sliding, rotating and/or flipping. One way to describe this is that if you were to cut your triangle out of a piece of paper, and a group mate did the same, then if you could pick up your triangle and set it down on the other triangle so that it overlapped exactly, then they would be congruent. Another way to describe this is if one student lists the side lengths and angle measures in order around the triangle, they should match up, in the same order with a congruent version. The congruent version may start in a different location, and may go clockwise instead of counterclockwise.

This paragraph foreshadows question 5 on Part II. The green point can be translated to the origin, the triangle can then be rotated until the red point is on the x-axis. If necessary, the triangle can be flipped.

To assist you, there are five Geogebra applets available. Each has its own strengths and weaknesses, and a given applet may not be appropriate for a given task, however it is best if you try all five applets and discuss the difficulties that arise.

Students should become expert on one of the applets. It takes a long time if each student uses each applet. It is also best if students don't do the problems in Part I in order, but instead pick the cases that are best suited for their tool. A geometric review of the various cases may be useful. In fact, you may wish to begin with question 1 from Part II to understand the cases.

The goal in each case is to use the Geogebra applet to create the triangle with the three given parts in as many ways as possible. You want to do things differently that the others in the group. If someone puts the side of length 7 to the left of the side of length 5, try putting yours on the right.

In some cases, it will be difficult to get your triangle to have exact values. Get as close as you can, but don't be too concerned if your values are off by a few hundredths.

Geogebra Applets

- Free Form Triangle

  This applet has six slider bars to control the x- and y-coordinates of three points. You may change the location of a point by adjusting the slider bar. If you need to make small changes you may click on the button on the slider bar and then use the arrow keys to increment by 0.01. You may also change the location of a point by dragging the point directly. This allows you to move the three points anywhere you want and therefore has the most flexibility. It may be able to find solutions that the others miss.

  Link: https://www.geogebra.org/m/mjqxpngj

  Try out these applets beforehand so you know how they work.

- Cartesian Restricted Triangle

  This applet has one point (A, in green) fixed at the origin and a second point (C, in red) fixed on the positive side of the x-axis with a red slider for the x-coordinate. A third point (B, in blue) is free to move anywhere with two sliders. The two blue sliders for B control the x- and y-coordinates.

  Link: https://www.geogebra.org/m/pftmepfm

  This is the natural way to control point B.

- Polar Restricted Triangle

  This applet is similar to the Cartesian Restricted Triangle with one point fixed at the origin, a second fixed on the positive side of the x-axis with a slider for the x-coordinate, and a third blue point (B). In this case, the third point is controlled by two sliders, one of which (length$C$, in red) determines the distance from the origin and the other (angle$A$, in green) controls the direction. Angles are defined as they are for the unit circle, $0°$ pointing to the right, and other angles being measured counterclockwise from the positive side of the x-axis.

  Link: https://www.geogebra.org/m/e3nhcb9f

  The SAS case is a perfect fit for this. It is worth asking students at the beginning which problems are easiest to do using this tool.

- Swinging Gate 1

  This applet has one point fixed at the origin, and a point (B, in blue) controlled with the length/direction (polar) controls (length$C$, angle$A$, in red). One side (c, in red) of the triangle connects those two points (A and B). The x-axis (or portion thereof) will be a second side of the triangle. The third side of the triangle (a, in green) swings from the second point (B). There are sliders to control the third side's length and direction (length$A$, direction$B$to$C$). Increasing the value of length$A$ will extend the green side, moving point C further from point B. You are not able to drag point C directly, you can only move point C by moving the slider bars. Again, the direction follows the unit circle convention, where the direction of point C relative to point B is measured counterclockwise from the positive side of the x-axis. For example, when the slider bar has direction$B$to$C = 0°$, point C will be to the right of point B. Since this third side hangs down from point B, the angle control is negative. The length of the bottom blue side can be found from the x-coordinate of C. The angle C won't be shown since most of the time, the third side doesn't complete a triangle.

  Link: https://www.geogebra.org/m/jusbrdx3

  This is designed for the SSA case.

- Swinging Gates 2

  The applet has one side (b, in blue) on the positive side of the x-axis. One point (A) is fixed at the origin, and the other (C, in red) floats on the positive side of the x-axis. There is a blue slider (length$B$, in blue) that controls the length of this side. The other two sides can swing above the x-axis. A triangle is formed when point B_1 and B_2 coincide. The lengths of these two sides (a and c) are controlled by sliders. Slider c controls the length of the red side. Slider a controls the length of the green side. Slider 'angle$A$' controls the angle (A) between the red side and the blue side. The slider labelled 'ExtAngleC' controls the exterior angle (outside the triangle) between the green side and the portion of the x-axis to the right of point C.

  Link: https://www.geogebra.org/m/cepf5cbq

  This is useful in the SSS case that violates the triangle inequality.

  At the end, it is worthwhile to ask students which applets were best suited for particular questions.

Part I

1. (SSS) Find the three angles of a triangle with side lengths 5, 7, and 8

   How many different triangles can be produced? What are the angles?

2. (SSS) Find the three angles of a triangle with side lengths 3, 4, and 8

   How many different triangles can be produced? What are the angles?

   This violates the triangle inequallity. Have students give some explanation as to why this won't work.

3. (AAA) Find the three sides of a triangle with angle measures 52°, 85° and 43°.

   How many different triangles can be produced? What are the side lengths?

   The angles determine the shape, but the triangle can be any size. Have students describe what is going on here.

4. (SAS) Find the missing side and two missing angles of a triangle where one angle of the triangle has measure 73° and the sides on either side of the 73° angle have lengths 4 and 6.

   How many different triangles can be produced? What is the length of the missing side? What are the measures of the missing angles?

5. (ASA) Find the missing angle and two missing sides of a triangle where two angles of the triangle have measures 84° and 63°, and the side between the two angles has length 5.

   How many different triangles can be produced? What are the lengths of the missing sides? What is the measure of the missing angle?

6. (AAS) Find the missing angle and two missing sides of a triangle where two angles of the triangle have measures 42° and 77°, and the side across from the 77° angle has length 9.

   How many different triangles can be produced? What are the lengths of the missing sides? What is the measure of the missing angle?

   Make sure students get things lined up correctly. Students may already be calculating the third angle before using the applet.

7. (SSA) Find the missing side and two missing angles of a triangle where one angle of the triangle has measure 52°, a side adjacent to the 52° angle has length 8 and the side across from the 52° angle has length 5.

   How many different triangles can be produced? What is the length of the missing side? What are the measure of the missing angles?

   This leg is too short. There are no answers.

8. (SSA) Find the missing side and two missing angles of a triangle where one angle of the triangle has measure 52°, a side adjacent to the 52° angle has length 8 and the side across from the 52° angle has length 6.4.

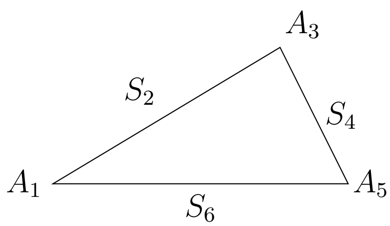
   This leg is too long and misses on the left side, therefore there is just one answer.

9. (SSA) Find the missing side and two missing angles of a triangle where one angle of the triangle has measure 52°, a side adjacent to the 52° angle has length 8 and the side across from the 52° angle has length 6.4.

   This triangle is just right, and has two answers.

Part II

1. A triangle has six parts that we are interested in, three side lengths and three angles. How many ways are there to choose three of the six parts? This question can be interpreted in two ways: one way is to think about the parts being labelled in order clockwise around the triangle, $A_1, S_2, A_3, S_4, A_5, S_6$,



   and then choosing three of the six. This has an answer using combinations/Pascal's triangle; the other way is to think of the relationship of the parts to one another, which has a connection to geometric proofs of congruence.

   This is 6 choose 3, which has 20 possibilities that can be broken up into the following categories:

   SSS (one case - subscripts 246)

   AAA (one case - subscripts 135)

   SAS (three cases - subscripts 234, 456, 612)

   ASA (three cases - subscripts 123, 345, 561)

   SSA (six cases - subscripts 241, 245, 463, 461, 623, 625)

   AAS (six cases - subscripts 134, 136, 352, 356, 512, 514)

2. When is it possible to solve an SSS triangle? When is it not possible?

   In Part I, what happened in question 2?

3. When will an SSA triangle have exactly one solution?

   Students here may have noticed that if $a > c$, then there will be only one solution.

4. When will an SSA triangle have no solutions?

   The simple answer is when side a is too short. Can they find a formula that defines 'too short'? $\boxed{a < c \cdot \sin A}$

5. If you can produce a triangle using the Free Form Triangle, can you produce a congruent triangle using the Polar Restricted Triangle? Discuss how you could slide, rotate, and flip a Free From Triangle until it meets the constraints of the Polar Restricted Triangle. Recall that the restrictions are that point A is at the origin, point C is on the positive side of the x-axis and point B is above the x-axis.