The goal of this worksheet is to learn how to modify quadratic equations in order to write them as a perfect square. Some students may struggle with believing the factoring of some polynomials is what it is. Whenever a student doubts their answer, encourage them to multiply it out and compare their answer to the polynomial they were supposed to factor.

1. Expand

   (a) $(x+1)(x+1)$ $\boxed{= x^2 + 2x + 1}$
   Students should be encouraged to write out the intermediate FOIL step if necessary.

   (b) $(x+4)^2$ $\boxed{= x^2 + 8x + 16}$

   (c) $2(x-3)^2$ $\boxed{= 2x^2 - 12x + 18}$
   Students should be encouraged to write out the intermediate steps.

   (d) $a(x-h)^2 + k$ $\boxed{= ax^2 - 2ahx + (ah^2 + k)}$
   You may need to tell students that we are treating x as the variable, and a, h and k as constants. Ask students 'What is the coefficient on x?' 'What is the constant?'

2. Factor

   (a) $x^2 + 6x + 9$ $\boxed{= (x+3)(x+3)}$

   (b) $2x^2 - 8x + 8$ $\boxed{= 2(x-2)(x-2)}$
   Watch to make sure students are factoring out the greatest common factor first.

3. Supply the missing constant so that the following functions are perfect squares. Write each function in both the expanded form and the factored form.

   (a) $f(x) = x^2 - 8x + \boxed{16}$ $\quad f(x) = (x-4)(x-4)$

   Ask students how they know which number to fill in. Pay attention to whether they are starting with the missing constants or with the factored form. Students may approach this in different ways, and it is good for them to see multiple perspectives.

   (b) $g(x) = x^2 - 7x + \boxed{\dfrac{49}{4}}$ $\quad g(x) = (x - \frac{7}{2})(x - \frac{7}{2})$

   Sometimes, students think that if a math problem contains a number that isn't clean, then they must have done it wrong. Confirm that this isn't always the case.

   (c) $h(x) = x^2 - \dfrac{7}{3}x + \boxed{\dfrac{49}{36}}$ $\quad h(x) = (x - \frac{7}{6})(x - \frac{7}{6})$

   By now students should be verbalizing the process.

   (d) $j(x) = 3x^2 - 7x + \boxed{\dfrac{49}{12}}$ $\quad h(x) = 3(x - \frac{7}{6})(x - \frac{7}{6})$

   Students should be encouraged to write out the steps of the process. Note that the order that they write things may not be in order on paper. Have students be explicit about the order they are doing things. When writing a polynomial in the standard form of a parabola, students may forget to add the constant to both sides. For those who remember to do so, ask them why this is necessary.

   (Hint: Factor out the 3 first, then supply the missing constant, then distribute the 3 back in.)

4. Comparing the factored form $f(x) = A(x+m)^2$ to the expanded form $f(x) = ax^2 + bx + c$, what is the relationship between A, m and a, b, c.

   This is a generalization of question 3 and refers to question 1d), which students may have recognized as the standard form of a parabola. Have student expand $A(x+m)^2$ and equate coefficients.

   $\boxed{a = A;\ b = 2Am;\ c = Am^2}$

5. If you begin with $g(x) = ax^2 + bx$, how do you find c so that $f(x) = ax^2 + bx + c$ will factor as a perfect square?

   You can refer students to question 3, particularly 3d). You can have students work further examples to find a pattern. There is also a connection to question 4.

6. Find the vertex of the following parabola by moving the constant to the left side, completing the square on the right side, adding the same constant to both sides, then writing in standard form $y = a(x-h)^2 + k$.

$$y = x^2 + 6x + 3$$

$$\boxed{y - 3 + 9 = x^2 + 6x + 9 \implies y = (x+3)^2 - 6}$$

Students will need to show more steps and talk through the process several times to prepare for the next problem. Do not fear the variables in the next question. Ask students to explain what they did in this problem, and reveal to them that they can apply the exact same technique there.

7. Find the vertex of the following parabola by moving the constant to the left side, completing the square on the right side, adding the same constant to both sides, then writing in standard form $y = a(x-h)^2 + k$.

$$y = x^2 + bx + c$$

$y = x^2 + bx + c \implies y - c = x^2 + bx$. We will want factors of $(x + \frac{b}{2})$, so we need to supply the missing constant to wind up with $y - c + \boxed{} = (x + \frac{b}{2})^2$. We can accomplish this by adding $\dfrac{b^2}{4}$ to both sides so that the right hand side factors. Therefore, we have $y - c + \dfrac{b^2}{4} = x^2 + bx + \dfrac{b^2}{4} = \left(x + \dfrac{b}{2}\right)^2$. Finding a common denominator on the left, we get $y + \dfrac{b^2 - 4c}{4} = \left(x + \dfrac{b}{2}\right)^2 \implies y = \left(x + \dfrac{b}{2}\right)^2 + \dfrac{4c - b^2}{4}$. Therefore the vertex is at $\left(-\dfrac{b}{2}, \dfrac{4c - b^2}{4}\right)$

8. Find the vertex of the following parabola by moving the constant to the left side, dividing by the coefficient on x^2, completing the square on the right side, adding the same constant to both sides, finding a common denominator on the left side, simplifying, then writing in standard form $y = a(x-h)^2 + k$.

$$y = 2x^2 + 5x + 3$$

Alternate approach: Find the vertex of the following parabola by moving the constant to the left side, factoring out the coefficient on x^2, completing the square on the right side, distributing the constant back in on the right hand side, adding the same constant to both sides, then writing in standard form $y = a(x-h)^2 + k$.

$$y = 2x^2 + 5x + 3$$

$\boxed{\left(-\frac{5}{4}, -\frac{1}{8}\right)}$ The process for questions 8, 9, 10 is identical to the process for question 7. This is a fairly advanced problem and not all students need to get this far. You can complete the square to arrive at the standard equation of a circle without getting to question 8.

9. Find the vertex of the following parabola. $y = ax^2 + bx + c$

$\boxed{\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)}$

10. Solve the equation

$$ax^2 + bx + c = 0$$

by moving the c to the right side, dividing by a, completing the square, adding the same constant to both sides, taking a square root, and subtracting the constant term on the left to get x by itself.

This should yield the quadratic formula. This is a very advanced question and it is expected that only the top students will do this.