Trigonometry

Activity 3c - Angle Sum Formulas Part II Instructor's Guide

The goal of this worksheet is to derive the angle-sum and angle-difference formulas for cosine (completed on previous worksheet), sine, and tangent. Students should be prepared to pay careful attention simplifying fractions during this activity.

1. In Part I, we developed the following formulas:

$$\begin{aligned}\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)\\ \cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)\end{aligned}$$

   Use the first formula to find a formula for

$$\begin{aligned}\cos(90^\circ+\theta) &= \cos(90^\circ)\cos(\theta) - \sin(90^\circ)\sin(\theta)\\ &= 0\cdot\cos(\theta) - 1\cdot\sin(\theta)\\ &= -\sin(\theta)\end{aligned}$$

   Check to see that students understand the substitutions being used. This is also a good chance to test the students' memory of unit circle values.

2. It is also true that $\sin(90^\circ+\theta) = \cos(\theta)$.

   This problem can be skipped.

   There are multiple ways to show this. Visually is one effective way. If you draw an arbitrary angle on the unit circle, and then that angle plus 90°, the cosine of the first will be the sine of the second. The same technique can be used for question 1. Another approach is to write $\cos(90^\circ+\theta-90^\circ)$, but don't suggest this to students. It is a good exercise in creativity to see if they can come up with something.

3. Let $\theta = M+N$.

   (a) Use the formula for $\cos(90^\circ+\theta)$ to show that $\sin(M+N) = -\cos(90^\circ+M+N)$.

$$\begin{aligned}\cos(90^\circ+\theta) &= -\sin(\theta)\\ \cos(90^\circ+(M+N)) &= -\sin(M+N)\\ -\cos(90^\circ+M+N) &= \sin(M+N)\end{aligned}$$

   Some students may be used to two-column proof from a high school geometry class. Whether they are familiar with two-column proof or not, get students used to supplying reasons for each step.

   Students may find this problem tricky because there are 3 terms in the input now. It may help to group them in parentheses, e.g., instead of $\sin(90+M+N)$, consider $\sin(90+(M+N))$ so that it's clear which variables are playing the roles of A, B, and theta from the original equations. The purpose of this step is to start to develop a formula for $\sin(M+N)$ using the angle-sum formula for cosine and the properties of complementary angles.

(b) Use the formula for $\cos(A+B)$, letting $A = 90° + M$ and $B = N$ to write a formula for $-\cos(90° + M + N)$.

$$-\cos(90° + M + N) \quad = \quad -\cos(90° + M)\cos(N) + \sin(90° + M)\sin(N)$$

This problem requires careful care with plugging numbers in. I suggest using parentheses to contain the argument of trig functions throughout the entire activity, even when the argument is unambiguous, to get students accustomed to the idea that the argument of a trig function cannot be easily freed from it. Some students may already make the connection to what is asked in question 3cd) and do the work here. It is not necessary to separate the questions in that way.

(c) Simplify the previous expression using the formulas found in problems 1 and 2.

$$\begin{aligned} -\cos(90° + M + N) &= -\cos(90° + M)\cos(N) + \sin(90° + M)\sin(N) \\ &= \sin(M)\cos(N) - \cos(M)\sin(N) \end{aligned}$$

(d) Combine problems 3a, b, c to derive a formula for $\sin(M + N)$.
$-\cos(90° + M + N) = \sin(M + N)$ by part 3a). Plugging this in to the last equation in part 3c) gives the answer.

$$\sin(M + N) = \sin(M)\cos(N) + \cos(M)\sin(N)$$

4. In problem (3) of the previous activity, you used a process to convert the formula for $\cos(A-B)$ to a formula for $\cos(A+B)$. Use this same process to convert the formula for $\sin(M+N)$ to a formula for $\sin(M-N)$.

If students did not fully understand this step in the previous activity, now it a good time to review it. A simple example might be, "We wish to write the difference of two numbers as the sum of two numbers, in order to be able to apply the angle-sum formula for sine. I want to write, say, $5-4$ as the sum of two numbers. What number completes the blank? $5 - 4 = 5 + \_\_\_\_$." Let $L = -N$ (there are many choices of substitution to make here).

$$\begin{aligned} \sin(M + L) &= \sin(M)\cos(L) + \cos(M)\sin(L) \\ \sin(M + -N) &= \sin(M)\cos(-N) + \cos(M)\sin(-N) \\ \sin(M - N) &= \sin(M)\cos(N) + \cos(M)\cdot(-\sin(N)) \\ &= \sin(M)\cos(N) - \cos(M)\sin(N) \end{aligned}$$

Now is also a good time to review the even/odd properties of functions.

5. In summary:

$$\begin{aligned}
\cos(A-B) &= \cos(A)\cos(B)+\sin(A)\sin(B)\\
\cos(A+B) &= \cos(A)\cos(B)-\sin(A)\sin(B)\\
\sin(A-B) &= \sin(A)\cos(B)-\cos(A)\sin(B)\\
\sin(A+B) &= \sin(A)\cos(B)+\cos(A)\sin(B)
\end{aligned}$$

6. Find formulas for $\tan(A-B)$ and $\tan(A+B)$ as follows:
   (a) Write $\tan(A-B)$ as $\dfrac{\sin(A-B)}{\cos(A-B)}$, then use 5) to rewrite in terms of $\sin A$, $\sin B$, $\cos A$ and $\cos B$.
   (b) Divide both the numerator and denominator by $\cos(A)\cos(B)$ and simplify.
   (c) Rewrite in terms of $\tan(A)$ and $\tan(B)$.
   (d) Repeat the process for $\tan(A+B)$

As you work with students, take note of students often writing 'tan=sin/cos' (omitting the necessary arguments to the function). This is an extremely delicate exercise in simplifying fractions. Encourage students to use the white boards for this. Lookout for common denominators when adding fractions.

$$\begin{aligned}
\tan(A-B) &= \frac{\sin(A-B)}{\cos(A-B)}\\
&= \frac{\sin(A)\cos(B)-\cos(A)\sin(B)}{\cos(A)\cos(B)+\sin(A)\sin(B)}\\
&= \frac{\sin(A)\cos(B)-\cos(A)\sin(B)}{\cos(A)\cos(B)+\sin(A)\sin(B)} \times \frac{\frac{1}{\cos(A)\cos(B)}}{\frac{1}{\cos(A)\cos(B)}}\\
&= \frac{\left(\frac{\sin(A)\cos(B)-\cos(A)\sin(B)}{\cos(A)\cos(B)}\right)}{\left(\frac{\cos(A)\cos(B)+\sin(A)\sin(B)}{\cos(A)\cos(B)}\right)}\\
&= \frac{\left(\frac{\sin(A)\cos(B)}{\cos(A)\cos(B)}-\frac{\cos(A)\sin(B)}{\cos(A)\cos(B)}\right)}{\left(\frac{\cos(A)\cos(B)}{\cos(A)\cos(B)}+\frac{\sin(A)\sin(B)}{\cos(A)\cos(B)}\right)}\\
&= \frac{\frac{\sin(A)}{\cos(A)}-\frac{\sin(B)}{\cos(B)}}{1+\tan(A)\tan(B)}\\
&= \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)}
\end{aligned}$$

Recalling that tangent is an odd function ($\tan(-\theta) = -\tan(\theta)$), we get:

$$\begin{aligned}
\tan(A+B) &= \tan(A-(-B)) \\
&= \frac{\tan(A)-\tan(-B)}{1+\tan(A)\tan(-B)} \\
&= \frac{\tan(A)-(-\tan(B))}{1+\tan(A)(-\tan(B))} \\
&= \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}
\end{aligned}$$