The goal of this worksheet is to derive, completely by hand, a formula for $\cos(A - B)$ and $cos(A+B)$. This is the first time students will be deriving a formula to use instead of being given a formula. As with the Piston Motion activity, one of the challenges of this worksheet is working with arbitrary variables. Students should be prepared to pay careful attention to FOILing expressions and substituting variables during this activity.

1. Find the following distances:

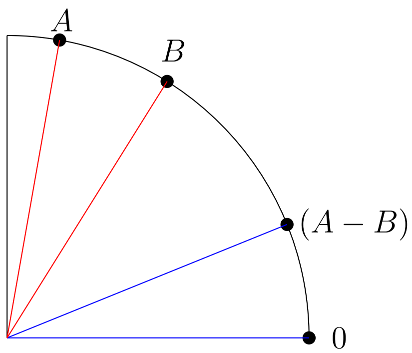
   (a) Find the distance between $(4, 0)$ and $(-2, 0)$
   (b) Find the distance between $(4, 3)$ and $(-2, 3)$
   (c) Find the distance between $(4, 3)$ and $(4, 6)$
   (d) Find the distance between $(-2, 3)$ and $(4, 6)$
   (e) Find the distance between $(-2, 3)$ and (h, k)
   (f) Find the distance between (x, y) and (h, k)

   These should all be applications of the formula $d = \sqrt{(x-h)^2 + (y-k)^2}$. It is common for one table of students to use the formula for slope instead. If you see a student doing this, let them! If they list the slope formula for part (f), ask them to re-apply that formula to 1(a) and see what goes wrong. Some students may realize that this is yet another application of the Pythagorean theorem! If they do, encourage them to explain the connection to their tablemates.

   On part a) lead students to the fact that you could just count the steps, and that subtraction is what finds the distance. Ask what is different in part b) (Nothing is different, it's just higher, but the y-value doesn't change, just the x-value.) Ask what is different in part c) (Now the x-values are staying the same and the y-values are changing. It is good to have students draw a picture once they get to part d). Something involving the Pythagorean Theorem should be discussed.

   This first question is really just a review of the distance formula.

2. Let A and B be first quadrant angles with $A > B > (A - B) > 0$.



(a) In the picture above, label the x- and y-coordinates of the points associated with A, B, $A - B$, and 0.

(Hint: the ordered pair for the angle 0 is $(1, 0)$, from the unit circle. The others will involve *sin* and *cos*.)

Each angle θ should have corresponding points of the form $(\cos(\theta), \sin(\theta))$. It is important to let students struggle with this! This activity is a very critical one for ideas to "click" with students. If they don't see the connection between sin and cos and the coordinates just yet, suggest they draw triangles on top of their drawings. Within these new triangles, apply the Pythagorean Theorem and/or SohCahToa. Using a specific value, like 60° for angle B may help start the conversation.

(b) Find a formula for the distance between the points associated with the angles A and B.

$\boxed{\sqrt{(\cos(B) - \cos(A))^2 + (\sin(B) - \sin(A))^2}}$ Students can be prompted that they wrote a similar formula in 1f). Whenever going over this, I highly encourage you to use parentheses around the arguments of trig functions, even when it's not ambiguous. This will get students in the habit of paying attention to what belongs inside of the parentheses, such that there is no ambiguity between $\sin(A - B)$ and $\sin A - B$ in the next problem.

(c) Find a formula for the distances between the points associated with angles $A - B$ and 0.

$$\boxed{\sqrt{(\cos(A - B) - \cos(0))^2 + (\sin(A - B) - \sin(0))^2}}$$
$$\boxed{= \sqrt{(\cos(A - B) - 1)^2 + (\sin(A - B))^2}}$$

(d) Explain why the distances found in part c) and d) are the same. There are many ways to explain this. Shelly's personal favorite is something along the lines of "$A - B$ is the same as $A - B - 0$." Mike likes some explanation about the sector from 0 to $A - B$ and the sector from A to B being the same. 'They are the same slice of pizza'.

(e) Set the distances in part c) and part d) equal to each other and simplify. Your result should be a formula for $\cos(A-B)$.

Use an ENTIRE board for this one, and be careful! Students may need to rewrite this after sketching out some things. Ask students about how they are FOILing in the early steps. Ask students about how they handled/simplified $\cos^2(A-B)$. (You can't simplify it. $A-B$ is just the name of some angle.). Ask students what else they recognize. ($\cos^2(A)+\sin^2(A)=1$).

$$\sqrt{(\cos(B)-\cos(A))^2+(\sin(B)-\sin(A))^2}=\sqrt{(\cos(A-B)-1)^2+(\sin(A-B))^2}$$

$$(\cos(B)-\cos(A))^2+(\sin(B)-\sin(A))^2=(\cos(A-B)-1)^2+(\sin(A-B))^2$$

$$\cos^2(B)-2\cos(A)\cos(B)+\cos^2(A)+\sin^2(B)-2\sin(A)\sin(B)+\sin^2(A)=$$
$$\cos^2(A-B)-2\cos(A-B)+1+\sin^2(A-B)$$

$$\cos^2(A)+\sin^2(A)+\cos^2(B)+\sin^2(B)-2\cos(A)\cos(B)-2\sin(A)\sin(B)=$$
$$\cos^2(A-B)+\sin^2(A-B)-2\cos(A-B)+1$$

$$1+1-2\cos(A)\cos(B)-2\sin(A)\sin(B)=1-2\cos(A-B)+1$$

$$2-2\left(\cos(A)\cos(B)+\sin(A)\sin(B)\right)=2-2\cos(A-B)$$

$$-2\left(\cos(A)\cos(B)+\sin(A)\sin(B)\right)=-2\cos(A-B)$$

$$\cos(A)\cos(B)+\sin(A)\sin(B)=\cos(A-B)$$

3. Let $C=-B$.

   That is, let C be the point on the unit circle in the fourth quadrant which has the same x-coordinate of B, but the opposite y-coordinate so that $C=-B$. The angle $A-C$ will be used in the formulas, but it is somewhat difficult to visualize.

   (a) How does $\cos(C)$ relate to $\cos(B)$? How does $\sin(C)$ relate to $\sin(B)$?
   Because cosine is an even function, $\cos(C)=\cos(-B)=\cos(B)$. Because sine is an odd function, $\sin(C)=\sin(-B)=-\sin(B)$. It is good to have students add this to the picture in question 2.)

   (b) Using your answer to 2e), write the formula for $\cos(A-C)$.
   $\boxed{\cos(A-C)=\cos(A)\cos(C)+\sin(A)\sin(C)}$

   (c) In 3b), replace C with $-B$.
   $\boxed{\cos(A--B)=\cos(A)\cos(-B)+\sin(A)\sin(-B)}$ Notice how I left the argument as $A--B$ at first instead of $A+B$. You may choose to describe this process as "writing the sum of two numbers as a difference of numbers," or something along those lines.

   (d) Using 3a), simplify the equation in 3c). Your result should be a formula for $\cos(A+B)$.
   $\boxed{\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)}$ Note the '$-$' sign factored to the front of that term. Keep an eye out for students who are *setting* C equal to $-B$ versus those who are *defining* C to be $-B$.