Combinations

University of Minnesota

Preliminaries and Objectives

Preliminaries

- General Counting Principle
- Permutations
- Factorial Notation

Objectives

- Count the number of ways to select k objects from a set of size n

Permutations

The number of ways to place k objects in order from a set of size n is

$$P(n,k) = \underbrace{(n)(n-1)(n-2)\dots(n-k+1)}_{k \text{ terms multiplied}} = \frac{n!}{(n-k)!}$$

Permutation Example

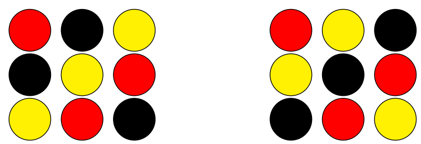
From a collection of five colored marbles, in how many ways, can you place three marbles in order?

$$\underline{\;5\;} \times \underline{\;4\;} \times \underline{\;3\;} = 60 = \frac{5!}{(5-3)!}$$

What if we only care about the colors and not the order?

Combinations

From a collection of five colored marbles, in how many ways, can you place three marbles in order?



For each combination of three colors, there are 3! permutations of that one combination.

Combinations

In general, for each combination of k objects, there are $k!$ permutations of that one combination.

Therefore,

$$P(n,k) = k! \cdot C(n,k)$$

and

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

where $C(n,k)$ denotes the number of combinations of k objects picked from a set of size n,

Other Notations

$$C(n,k) = {}_nC_k = \binom{n}{k}$$

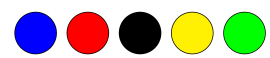
Connection to Pascal's Triangle

$\binom{n}{k}$ is the k^{th} entry in row n of Pascal's Triangle, where the n^{th} row begins

$$1 \quad n \quad \dots$$

and the index k begins at zero, so the 0^{th} term of row n is 1, the first term is n etc.

Connection to Pascal's Triangle - Example



1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

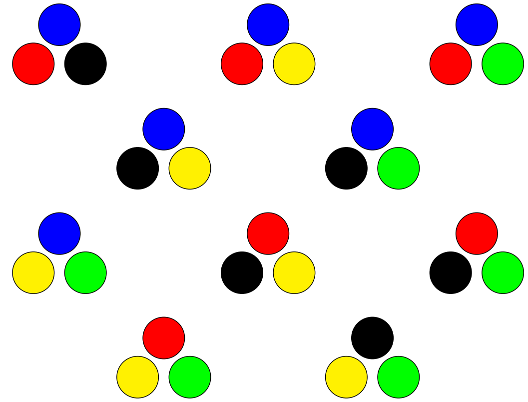
How many ways are there to select a combination of three marbles from a set of five?

Connection to Pascal's Triangle - Example

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The number of ways to select a combination of three marbles from a set of five is 10.

Connection to Pascal's Triangle - Example



Combinations and Pascal's Triangle

The recursion for Pascal's Triangle is

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

This also works for combinations as in this example:

How many combinations of 3 marbles can be made from a set of 5 marbles?

How many combinations of 3 marbles can be made from a set of 5 marbles that contain the blue marble? ${}_4C_2 = 6$
How many combinations of 3 marbles can be made from a set of 5 marbles that don't contain the blue marble? ${}_4C_3 = 4$