1. Combinations

2. You should be familiar with the General Counting Principle and permutations, including the factorial notation to compute answers to permutation questions. In this lesson, we will count the number of ways to choose k objects when we are not concerned with the order of selection.

3. Recall that when we are placing k objects in order, we multiply k terms, beginning with n, and going down by one each time.

4. (a) For example, to put three colored marbles in order, we multiply the three numbers, 5 times 4 times 3.
   (b) But what if we don't care about the order, only the colors chosen?
   (c) Here is one possible choice, red, black, yellow. These three colors make six permutations, but it is just one combination of colors.
   (d) For each combination of three colors, there are 3! permutations.

5. In general, for each combination of k objects, there are $k!$ ways to rearrange those objects into a permutation, so to get the number of permutations, we can multiply the number of combinations, by the number of permutations for each combination. Since we have a formula for counting permutations, we can get a formula for combinations by dividing by $k!$.

6. Similar to permutations, there are several notations for combinations, substituting a 'C' for combination in place of the 'P' for permutation. The final notation is called the binomial coefficient. It places the n and k inside parentheses, stacked as if it were a fraction, except without the fraction bar. It is read 'n choose k'.

7. The reason it is called the binomial coefficient is it's connection to Pascal's triangle, which among other things, determines the coefficients in the expansion of a binomail like $(x + y)^n$. n choose k is the k^{th} entry in row n of Pascal's Triangle, where the n^{th} row begins 1, n ... and the value for k begins at 0, and the proceeds to the right, so that when $k = 0$, $\binom{n}{k}$ is 1, when $k = 1$, $\binom{n}{k}$ is n, and so on.

8. (a) Here is an example. How many ways are there to select a combination of three marbles from a set of five, when we don't care about the order? To find the answer, we need to find the fifth row, since we are choosing from five total marbles. The fifth row begins 1, 5 ...
   (b) In the fifth row, we now need to find the entry corresponding to $k = 3$. Beginning at the left with $k = 0$, we see the numbers 1, 5, 10, 10, 5, 1. These correspond to the k-values 0, 1, 2, 3, 4, and 5, so the entry highlighted in red is the value we need. 5 choose 3 is 10.

9. Here are the 10 combinations.

10. (a) To see that the recursion for Pascal's triangle also makes sense for combinations, we will divide our combinations into two categories. The recursion we are looking for is that to find an entry in row n of Pascal's triangle, we look at the previous row, row $(n-1)$ and add the entries that are to the left and to the right in the previous row. Let's use a similar idea to find the number of combinations of three marbles drawn from a set of five, if we know how to find the combinations in a set of four.

    (b) We begin by picking a special color, and then looking at combinations of the remaining four colors. If the three marbles contain the blue marble, we need to choose two of the remaining four, so one of the two pieces that we need to add together is 4 choose 2. Pick the blue marble, and then pick two of the remaining four. The other piece comes from those combinations that don't include blue, in which case, we need to choose three of the remaining four, which is 4 choose 3. Altogether, we get 5 choose 3 by adding 4 choose 2 and 4 choose 3. 4 choose 2 is the combinations that include blue and 4 choose three is the combinations that don't include blue.